REGULARITY FOR OBSTACLE PROBLEMS WITH APPLICATIONS TO THE FREE BOUNDARIES IN THE CONSTRAINED LEAST GRADIENT PROBLEM

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The *constrained least gradient problem* involves minimizing

$$\int_{\Omega} |\nabla u| \, dx$$

amongst all functions u defined on a given bounded set Ω in $\mathbb{R}^n$, satisfying given boundary values ϕ defined on $\partial\Omega$ and a gradient constraint $|\nabla u| \leq 1$ a.e. in Ω. This particular problem was considered by Kohn and Strang ([KS1], [KS2]) and they showed how such problems could arise if one tries to find a bar of constant cross-section which will support a given load and has lightest weight. This type of problem is non-convex and in most cases will not have a solution. However it is possible to convexify the integral leading to the constrained least gradient problem. Minimizing sequences for the original problem will be minimizing sequences for the new one and the infimum of values of the integral will be the same for each problem. The main advantage is that the convexified problem will have a solution.

For applications to the problem above, that is of finding a bar of lightest weight which will support a given load, there are several additional things which can be learnt from solutions to the relaxed problem. Firstly, constructing the solution and evaluating the corresponding integral gives the minimum possible weight. (Usually this weight can't be achieved but at least one can then test a proposed design to see if it is close to the optimal weight.) Secondly, in regions where $|\nabla u| = 0$, and so u is constant, there is no need for material in the rod. Thirdly, in regions where $|\nabla u| = 1$ the stress is at a maximum, the rod will behave plastically and there is no hope of weight reduction. Finally, in the region where $0 < |\nabla u| < 1$ one may hope to reduce weight by using some type of fibred design