ON SOME TRACE INEQUALITIES

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§1 INTRODUCTION

Let $A \geqq B \geqq 0$ be positive operators on a Hilbert space. It is well-known that this order assumption implies $Tr(f(A)) \geqq Tr(f(B))$, where Tr denotes the usual trace and f is a continuous increasing function on $\mathbb{R}_+$ with $f(0) = 0$. In fact, singular numbers $\{\mu_n(\cdot)\}_{n=1,2,\ldots}$ (see [6], [7] for details) satisfy

$$\mu_n(f(A)) = f(\mu_n(A)) \geqq f(\mu_n(B)) = \mu_n(f(B))$$

because of $\mu_n(A) \geqq \mu_n(B)$ (a consequence of the min-max expression for $\mu_n(\cdot)$). Hence, by summing up over n, one obtains the desired estimate.

The purpose of the present note is to point out two generalizations of the above mentioned trace inequality.

§2 RESULTS

Let A, B be positive operators on a Hilbert space H satisfying $A \geqq B \geqq 0$. By setting $q = 2$ in Furuta's inequality ([5]), we obtain

$$A^{(p+2r)/2} \geqq (A^r B^p A^r)^{1/2} \tag{1}$$

as long as $p, r \geqq 0$ satisfy

$$(1+2r)2 \geqq p + 2r, \quad \text{i.e.,} \quad 2 + 2r \geqq p. \tag{2}$$

Extending the continuous linear map

$$A^{(p+2r)/4}\zeta \in R(A^{(p+2r)/4}) \mapsto (A^r B^p A^r)^{1/4}\zeta \in H$$

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