# Subelliptic operators on Lie groups

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## 1 Introduction

Hörmander's analysis [Hör] of hypoelliptic operators lent impetus to the development of the theory of subelliptic operators on Riemannian manifolds or on Lie groups. Recent results have been surveyed in the article by Jerison and Sánchez-Calle [JSC], the lecture notes of Varopoulos, Saloff-Coste and Coulhon [VSC] or the books by Davies [Dav] and Robinson [Rob]. The Lie group theory, which has been extensively studied for subelliptic operators with real coefficients, concentrates on analyzing the heat semigroup generated by the operator and the corresponding heat kernel. In this review we describe the main results of this theory under slightly more general assumptions than hitherto.

Throughout the sequel we adopt the general notation of [Rob]. In particular $G$ denotes a $d$-dimensional Lie group which we may assume to be connected because all analysis takes place on the connected component of the identity. The Lie algebra of $G$ is denoted by $\mathfrak{g}$. Furthermore $(\mathcal{X}, G, U)$ is used for a continuous representation of $G$ on the Banach space $\mathcal{X}$ by bounded operators $g \mapsto U(g)$. Both strong and weak* continuity are considered. Moreover if $a_i \in \mathfrak{g}$ then $A_i$ denotes the generator of the one-parameter subgroup $t \mapsto U(e^{-ta_i})$ of the representation. The $C^n$-subspaces $\mathcal{X}'_n$ of the representation $(\mathcal{X}, G, U)$ with respect to a subbasis $a_1, \ldots, a_{d'}$ of $\mathfrak{g}$ is the common domain of all monomials $M_m$ of order $m \leq n$ in the generators $A_1, \ldots, A_{d'}$. The $C^n$-norm is defined by

$$\|x\|'_n = \sup_{0 \leq m \leq n} \|M_m x\|$$

where the supremum is over all the monomials and $M_0 = I$. The $C^\infty$-elements $\mathcal{X}'_\infty$ of the subbasis are then defined by

$$\mathcal{X}'_\infty = \bigcap_{n \geq 1} \mathcal{X}'_n \quad .$$

Similarly if $a_1, \ldots, a_d$ is a full vector space basis of $\mathfrak{g}$ we use the notation $\mathcal{X}_n$ and $\|\cdot\|_n$ for the corresponding $C^n$-subspace and norm and denote the $C^\infty$-elements by $\mathcal{X}_\infty$.

If the representation $(\mathcal{X}, G, U)$ is strongly continuous, $\mathcal{X}^*$ is the dual of $\mathcal{X}$ and $U(g)^*$ the adjoint of $U(g)$ then one has a dual representation $(\mathcal{X}^*, G, U_*)$, where

$$U_*(g) = U(g^{-1})^* \quad ,$$

which is weakly* continuous. Alternatively if $(\mathcal{X}, G, U)$ is weakly* continuous, $\mathcal{X}_*$ is the predual of $\mathcal{X}$ and $U(g)^*$ is the adjoint of $U(g)$ on $\mathcal{X}_*$ one has a dual representation $(\mathcal{X}_*, G, U_*)$ which is strongly continuous. We will denote both cases with the common notation $(\mathcal{F}, G, U_*)$ with $\mathcal{F} = \mathcal{X}^*$, or $\mathcal{X}_*$.

The theory of subelliptic operators is constructed from a Lie algebraic basis $a_1, \ldots, a_{d'}$ of $\mathfrak{g}$, i.e., a finite sequence of linearly independent elements of $\mathfrak{g}$ whose Lie algebra generates $\mathfrak{g}$. Thus there is an integer $r$ such that $a_1, \ldots, a_{d'}$ together with all commutators

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