UNITARY APPROXIMATION AND SUBMAJORIZATION

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0. Introduction

We begin by considering the following inequality for complex numbers:

$$\text{(i) } |x-1| \leq |x-u| \leq |x+1|, \quad \forall\, 0 \leq x \in \mathbb{R} \text{ and } u \in \mathbb{C} \text{ with } |u|=1,$$

and the equivalent inequality:

$$\text{(ii) } |\,|z|-1\,| \leq |vz-1| \leq |\,|z|+1\,|, \quad \forall\, z \in \mathbb{C} \text{ and } v \in \mathbb{C} \text{ with } |v|=1.$$

It has been shown by Ky Fan and A. J. Hoffman [FH] that the inequality (i) remains valid if x is replaced by a given $n \times n$ Hermitian positive semi-definite matrix, u by any $n \times n$ unitary matrix and the modulus of a complex number is replaced by a unitarily invariant norm. Subsequently (ii) was shown to hold by D. J. van Riemsdijk [vR] for a certain class of symmetric norms, with z a bounded linear operator on a separable Hilbert space H, v any partial isometry with initial space containing the range of z. It is a well-known fact, due to Ky Fan [Fa], that metric inequalities in symmetrically normed ideals of compact operators are consequences of corresponding submajorization inequalities for singular values, and this indeed is the approach of [FH]. The approach of [vR] is based on an extension to arbitrary bounded linear operators of the notion of singular value sequence of a compact operator given in the monograph of Gohberg and Krein [GK], and again the metric inequalities given in [vR] are derived from corresponding submajorization inequalities. We mention further that special cases of the results of [vR] have also been given, in the setting of the Schatten p−classes and using methods of independent interest, by Aiken, Erdos and Goldstein [AEG], to which the reader is referred for an illuminating discussion of the relation of such inequalities to quantum chemistry.

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