SOME REMARKS ON OSCILLATORY INTEGRALS

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1. Introduction

The purpose of this note is to describe some results about oscillatory integral operators. Specifically we are interested in bounds in Lebesgue spaces of operators given by

$$T_\lambda f(x) = \int_{\mathbf{R}^k} e^{i\lambda\varphi(x,\xi)}\ f(\xi)\ d\xi,$$

with $\varphi(x,\xi)$ a real-valued smooth function on $\mathbf{R}^n \times \mathbf{R}^k, k \leq n$. Obviously T_λ is bounded as maps from L^q_{comp} to L^p_{loc}. What is of interest here is the dependence of the norm for increasing λ. This will of course depend on the conditions we put on the phase function φ. To guarantee that φ lives on an open subset of $\mathbf{R}^n \times \mathbf{R}^k$ it is natural to start with the condition

$$\text{rank } d_\xi d_x \varphi = k, \quad x \in \mathbf{R}^n \text{ and } \xi \in \mathbf{R}^k. \tag{1}$$

We will assume this condition throughout this note. For work related to weaker assumptions see, e.g. [21] and [18]. One of the questions we will ask is: What is the optimal (q,p)-range for which the operator T_λ has norm of order $\lambda^{-n/p}$? In particular we would like to understand how this range will depend on k.

To put things in perspective let us begin by describing what is known for the case $k = n$: A model phase function here is $\varphi(x,\xi) = x \cdot \xi$, for $x, \xi \in \mathbf{R}^n$. Then T_λ is a localized version of the Fourier transform and the (L^q_{comp}, L^p_{loc})-boundedness properties are covered by the Hausdorff-Young inequality. For general phase functions satisfying (1) the L^2-theory of Fourier integral operators gives

$$|||T_\lambda|||_{L^q_{comp}\to L^p_{loc}} \leq\ C\ \lambda^{-n/p},$$

with $p = q' \geq 2$ the dual exponent of q, i.e. $1/q' + 1/q = 1$.

Next we consider the case $k = n-1$: A basic result was obtained by E. M. Stein in the sixties. He discovered, for $n \geq 2$, that the Fourier

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