SINGULARITIES AND THE WAVE EQUATION ON CONIC SPACES

RICHARD B. MELROSE AND JARED WUNSCH

Introducing polar coordinates around a point in Euclidian space reduces the Euclidian metric to the degenerate form

$$dr^2 + r^2\, d\omega^2 \tag{1}$$

where r is the distance from the point and $d\omega^2$ is the round metric on the sphere. If X is an arbitrary manifold with boundary, the class of *conic metrics* on X is modeled on this special case. Namely, a conic metric is a Riemannian metric on the interior of X such that for some choice of the defining function x of the boundary ($x \in \mathcal{C}^\infty(X)$ with $\partial X = \{x = 0\}$, $x \geq 0$, $dx \neq 0$ on ∂X), the metric takes the form

$$g = dx^2 + x^2 h \text{ on } X^\circ = X \backslash \partial X, \text{ near } \partial X.$$

Here h is a smooth symmetric 2-cotensor on X such that $h_0 = h|_{\partial X}$ is a metric on ∂X.

In fact a general conic metric can be reduced to a form even closer to (1) in terms of an appropriately chosen product decomposition of X near ∂X, that is, by choice of a smooth diffeomorphism

$$[0, \epsilon)_x \times \partial X \xrightarrow{F} O \subset X, \ O \text{ an open neighborhood of } \partial X\,. \tag{2}$$

The normal variable in $x \in [0, \epsilon)$ is then a boundary defining function, at least locally near ∂X, and the slices $F \restriction_{x=x_0}$ have given diffeomorphisms to ∂X. Now such a product decomposition can be chosen so that

$$F^* g = dx^2 + x^2 h_x, \text{ in } x < \epsilon, \tag{3}$$

where h_x is a family of metrics on ∂X.

This reduced form is closely related to the behavior of geodesics near the boundary. Up to orientation and parameterization there is a unique geodesic reaching the boundary at a given point p. In particular the normal fibration of X near ∂X given by the segments $F([0, \epsilon) \times \{p\})$, $p \in \partial X$, consists of geodesics which hit the boundary, each at the corresponding point p.

We shall discuss here the behavior of solutions to the wave equation

$$(D_t^2 - \Delta)u = 0 \text{ on } \mathbb{R} \times X^\circ \tag{4}$$