## 12. The simple theory of types

In order to avoid the logical paradoxes, Russell invented the theory of types. The idea is to distribute all objects of thought into different types or, in other words to assume that they can be put into different layers or at different levels. We have some original objects called objects of type 0 (or 1 if one prefers). Sets of these objects or relations between them are objects of type 1. Sets of these again are objects of type 2, and so on. Further, the membership relation $x \epsilon y$ shall only have a meaning, if $y$ is of type $n+1$ as often as $x$ is of type $n$. Composite propositional functions $\phi(x)$ built up from atomic propositions $x \epsilon y$ have then only meaning if it is possible to attach numbers to the occurring variables such that always the symbol $y$ in every occurring atomic proposition $x \epsilon y$ gets the number $n + 1$ when $x$ gets the number $n$. Such expressions $\phi(x)$ are called stratified.

We may now set up the following axiom of comprehension: For any stratified $\phi(x)$ there exists a $y$ such that the equivalence

$$x \epsilon y \leftrightarrow \phi(x)$$

is generally valid, that is, it is valid for all $x$ of type $n$ if $y$ is of type $n+1$. Since we do not introduce negative types, there will be a lowest possible type for $x$ in $\phi(x)$, say $n_0$. Then the axiom asserts

$$\text{(I)} \qquad (Ey)(x)(x \epsilon y \leftrightarrow \phi(x)),$$

where the range of the universal quantifier is the domain of all objects of type $n$, $n \geqq n_0$, and the range of $(Ey)$ is the domain of objects of type $n + 1$. The identity relation $x = y$ might be introduced as an undefined notion beside the membership relation $\epsilon$. Then we would have to set up the axiom

$$(x = y) \rightarrow (\psi(x) \leftrightarrow \psi(y))$$

for every stratified $\psi(x)$. It is simpler, however, to use only $\epsilon$ as an undefined notion and define $=$ by letting $x = y$ stand for the validity of the equivalence,

$$\psi(x) \leftrightarrow \psi(y)$$

for any stratified $\psi$. We then also need, however, the axiom of extensionality

$$\text{(II)} \qquad (z)(z \epsilon x \leftrightarrow z \epsilon y) \rightarrow (x = y).$$

It is seen at once that the axioms of the power set and the union in the Zermelo-Fraenkel theory are valid statements here, and also the axiom of separation for stratified $C(x)$. As to the axioms of the small sets, these are also valid with the restriction that $\{a,b\}$ can be built only when $a$ and $b$ are of the same type. It must be noticed, however, that we get not only universal sets of different types but also null sets of different types. Indeed

$$(Ey)\ (x \epsilon y \cdot v \cdot x \bar{\epsilon} y) \text{ and } (y)(x \epsilon y \cdot \& \cdot x \bar{\epsilon} y)$$

used as $\phi(x)$ in (I) define, if $y$ runs through all individuals of type $n + 1$, the universal set of type $n$ respectively the null set of type $n$.

Because of the restriction in building the set $\{a,b\}$, we ought to look at the union and intersection of two sets. If $A(x)$ and $B(x)$ are two stratified