9. The notions "finite" and "infinite"

We will now leave for a while the theory of transfinite numbers and deal with the notion "finite set". There are different possible definitions of this notion and with the aid of the well-ordering theorem they can be proved to be equivalent. Without the axiom of choice the proof of this equivalence seems impossible. I shall prove that the well-ordered finite sets are just the well-ordered sets that are also inversely well-ordered, that is, there is in every non-empty subset also a last element.

Definition of the notion inductive finite set:

A set u is inductive finite, if the following statement is true:

$$(x)(x\epsilon \mathfrak{U}\mathfrak{U}u \;\&\; (0\,\epsilon x) \;\&\; (y)(z)(y\,\epsilon x \;\&\; z\,\epsilon u \rightarrow y \;\cup\{z\}\,\epsilon x) \rightarrow u\,\epsilon x).$$

In ordinary language this means that every set x of subsets of u, such that $0\,\epsilon x$ and as often as $y\,\epsilon x$ and $z\,\epsilon u$, always $y \cup \{z\}\,\epsilon x$, contains u as element.

Remark. Such sets x of subsets always exist. Indeed Uu is such a set x.

According to this definition we of course have the following principle of induction: If a statement S is valid for 0, and S is always valid for $y \cup \{z\}$ if it is true for y, $y \subseteq u$, $z\,\epsilon u$, u inductive finite, then S is valid for u. I shall now prove a few theorems on the inductive finite sets.

Theorem 35. *If u is inductive finite, so is u* $\cup \{m\}$.

Proof. It suffices to assume $m\,\bar{\epsilon}\,u$. Let x be a set of subsets of $u \cup \{m\}$ such that $0\,\epsilon x$ and if $y\,\epsilon x$ and $z\,\epsilon u \cup \{m\}$ then $y \cup \{z\}\epsilon x$. Further, let x' be the subset of x consisting of all elements of x which are $\subseteq$ u. Then $0\,\epsilon x'$ and as often as $y\,\epsilon x'$, $z\,\epsilon u$, we have $y \cup \{z\}\,\epsilon x$ and therefore also $y \cup \{z\}\,\epsilon x'$. Thus, u being inductive finite, $u\,\epsilon x'$. But $u\,\epsilon x$ and $m\,\epsilon u \cup\{m\}$ yields $u \cup\{m\}\epsilon x$. Hence the theorem is correct.

Theorem 36. *Every subset of an inductive finite set u is inductive finite.*

Proof. Let v be $\subseteq$u. I consider the set x of subsets w of u such that $w \cap v$ is inductive finite. It is obvious that $0\,\epsilon x$, because the set 0 is inductive finite. Let y be ϵx and $z\,\epsilon u$. Then $\underline{y} \cap v$ is inductive finite and $(y \cup \{z\}) \cap v$ is either $y \cap v$, namely when $z\,\bar{\epsilon}\,v$, or $(y \cap v) + \{z\}$, namely if $z\,\epsilon v$. But by the preceding theorem also $(y \cap u) + \{z\}$ is inductive finite. Thus as often as $y\,\epsilon x$, $z\,\epsilon u$, we have $y \cup\{z\}\epsilon x$. Since u is inductive finite, it follows that $u\,\epsilon x$. Hence $u \cap v$ is inductive finite, that is, v is inductive finite.

It follows easily from this that each subset v of u, u inductive finite, must be an element of every set of subsets of the kind mentioned in the definition of inductive finiteness.

Theorem 37. *If u and v are inductive finite, so is u* $\cup$ *v.*

Proof. We consider the subset x of all subsets w of u such that $w \cup v$ is inductive finite. Obviously $0\,\epsilon x$. Let $y\,\epsilon x$ and $z\,\epsilon u$. By the previous theorem, y is inductive finite. Further $y \cup v$ is inductive finite so that $y \cup \{z\} \cup v$ is also inductive finite which means that $y \cup\{z\}\,\epsilon x$. Since u is inductive finite, $u\,\epsilon x$. This again means that $u \cup v$ is inductive finite.