8. Sets representing ordinals

There exists a class of sets of such a particular structure that they may suitably be said to represent ordinal numbers. I shall first mention the definition by R. M. Robinson (1937).

A set M is an ordinal, if

1) M is transitive. That a set M is transitive means that it contains its union. In symbols: $(x)(y)((x\epsilon y) \,\&\, (y \epsilon M) \rightarrow (x \epsilon M))$.
2) Every non empty subset N of M is basic, which means that it is disjoint to one of its elements. In logical symbols: $(Ex)(x \epsilon N \,\&\, (x \cap N = 0))$.
3) If $A \neq B$, $A \epsilon M$ and $B \epsilon M$, then either $A \epsilon B$ or $B \epsilon A$.

I shall call every set M with the properties 1), 2), 3) an R - ordinal.

Remark 1. If $\mathfrak{M}$ is a class of R - ordinals, then the intersection of all elements of $\mathfrak{M}$ is again an R - ordinal. Indeed, if M_0 is this intersection, we have that if $A \epsilon B$, $B \epsilon M_0$, then $A \epsilon B$, $B \epsilon M$ for every M in $\mathfrak{M}$, whence $A \epsilon M$ because M is transitive, whence $A \epsilon M_0$, because this is valid for every M in $\mathfrak{M}$. Thus M_0 is transitive. Let $0 \subset N \subseteq M_0$. Then for any M in $\mathfrak{M}$ we have $0 \subset N \subseteq M$, whence by 2) M_0 has the property 2). Finally let A and B be different and ϵM_0. Then for any M in $\mathfrak{M}$ we have A and $B \epsilon M$, whence by 3) either $A \epsilon B$ or $B \epsilon A$. Thus M_0 has the property 3).

Remark 2. Further it may be remarked that if M is an R - ordinal we have $M \bar{\epsilon} M$, because $M \epsilon M$ would mean that the subset $\{M\}$ of M was not basic.

Theorem 31. *Every R-ordinal M is the set of all its transitive proper subsets.*

Proof. Let C be ϵM. Since M is transitive, C must be $\subseteq M$. Indeed C is $\subset M$. $C = M$ is impossible, because that would mean $M \epsilon M$, which is impossible by Remark 2. Further C must be transitive. Indeed let $A \epsilon B$, $B \epsilon C$. Then $B \epsilon M$, whence $B \subseteq M$, whence $A \epsilon M$, whence $A \subseteq M$. By 3) we have either $A \epsilon C$ or $C \epsilon A$ or $A = C$. I assert that $C \epsilon A$ and $C = A$ are impossible. Indeed, $C \epsilon A$ would imply that $\{A,B,C\}$ is not basic, and $C = A$ would mean that $\{A,B\}$ is not basic. Hence $A \epsilon C$, that is, C is transitive. So far I have proved that every element C of M is a transitive proper subset of M.

Let, on the other hand, C be a transitive proper subset of M. Then $0 \subset M - C$ so that by 2) an element A of $M - C$ exists such that $A \cap (M - C) = 0$. Then, if $B \epsilon C$, neither $A = B$ nor $A \epsilon B$, because of the transitivity of C. Therefore $B \epsilon A$ and thus $C \subseteq A$ because $B \epsilon C$ yields $B \epsilon A$ for all B. Since $A \subseteq M$ and $A \cap (M - C) = 0$, it follows that $A \subseteq C$, whence $A = C$, whence $C \epsilon M$. Thus I have proved that every transitive proper subset of M is element of M.

Remark 3. It is clear according to this that every element of an R - ordinal is an R - ordinal.