2. Ordered sets. A theorem of Hausdorff.

One obtains a more complete idea of Cantor's work by studying his theory of ordered sets. As to the notion "ordered set" this is nowadays mostly defined in the following way:

A set M is ordered by a set $P \subseteq M^2$, if and only if the following statements are valid:

1) No pair (m,m), $m \in M$, is $\in P$.
2) For any two different elements m and n of M either $(m,n) \in M$ or $(n,m) \in M$ but not both at the same time.
3) for all $m,n,p \in M$ we have $(m,n) \in P$ & $(n,p) \in P \rightarrow (m,p) \in P$ (transitivity).

As often as $(m,n) \in P$, we also say m is less than n or m preceeds n, written $m < n$.

If M and N are ordered sets and there exists a one-to-one order-preserving correspondence between them, Cantor said that they were of the same order type and wrote $M \simeq N$. They are also called similar. Evidently two ordered sets of the same order type possess the same cardinal number; but the inverse need not be the case. Only for finite sets is it so that two ordered sets of the same cardinality are also of same type. Cantor denoted by $\overline{M}$ the order type of M.

That two infinite ordered sets possessing the same cardinal number may have different order types is seen by the simple example of the set of positive integers on the one hand and that of the negative integers on the other. Both sets are denumerable, but obviously not ordered with the same type, because the former has a first member, which the other has not, whereas the latter has a last member, which the former does not possess. Cantor studied to a certain extent the denumerable types, also types of the same cardinality as the continuum, but above all he studied the so-called well-ordered sets. In this short survey of Cantor's theory I shall only mention some of the most remarkable of his results and add a theorem of Hausdorff.

It will be necessary to define addition and multiplication of ordered sets. If A and B are ordered by P_A and P_B while A and B are disjoint, the sum set A + B will be ordered by $P_A + P_B + A \cdot B$. We have of course to distinguish between A + B and B + A. This addition may be extended to the case of an ordered set T of ordered sets A,B,C,... which are mutually disjoint. Indeed the union (or sum) ST will then be ordered by the sum of the sets P_A, P_B, P_C, ... and the products $X \cdot Y$ when (X,Y) run through all pairs which are the elements of the ordering set P_T for T.

By the product of two ordered sets A and B we understand $A \cdot B$ ordered lexicographically: that means that a_1, b_1 precedes a_1, b_2 if either a_1 precedes a_2, or $a_1 = a_2$, but b_1 precedes b_2. This definition also admits generalization, but that will not be necessary just now.

If a 1-1-correspondence exists between the ordered sets M and N such that the order is reversed by the correspondence, then $\overline{N}$ is said to be the inverse order type of $\overline{M}$. For example the order type of the set of negative integers is the inverse of the type of the positive integers. Cantor denotes the inverse of the order type α by α^*. Thus ω and ω^* denote the types of the sets of positive and of negative integers.