Chapter 7

THE FIRST APPROXIMATION THEOREM

1. The properties A_d, B, and C.

While the last two chapters depended on purely algebraic ideas, we now introduce real and g-adic algebraic numbers and study their rational approximations with respect to the corresponding absolute value or g-adic value, respectively. Here, as usual,

$$g = p_1^{e_1} \dots p_r^{e_r} \geq 2,$$

where $p_1, \dots, p_r$ are distinct primes, and $e_1, \dots, e_r$ are positive integers; the g-adic value $|A|_g$ of $A \leftrightarrow (\alpha_1, \dots, \alpha_r)$ is defined by

$$|A|_g = \max\left(|\alpha_1|_{p_1}^{\frac{\log g}{e_1 \log p_1}}, \dots, |\alpha_r|_{p_r}^{\frac{\log g}{l_r \log p_r}} \right).$$

The later occurring g'-adic and g''-adic values $|a|_{g'}$ and $|a|_{g''}$ are defined analogously.

The letter ξ always denotes a fixed real algebraic number, and the letter Ξ a fixed g-adic algebraic number. Only ξ satisfying

$$\xi \neq 0$$

and only $\Xi \leftrightarrow (\xi_1, \dots, \xi_r)$ satisfying

$$\xi_1 \neq 0, \dots, \xi_r \neq 0$$

will be considered. We denote by

$$F(x) = F_0 x^f + F_1 x^{f-1} + \dots + F_f, \qquad \text{where} \qquad f \geq 1,\ F_0 \neq 0,\ F_f \neq 0,$$

a polynomial of lowest degree with integral coefficients having either ξ, or Ξ, or both ξ and Ξ, as zeros; hence, by Chapter 3, $F(x)$ has no multiple factors. As before, we put

$$c = 2\max(|F_0|, |F_1|, \dots, |F_f|), \qquad \text{so that} \qquad c > 1.$$

Next we denote by

$$\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \dots\}$$

a fixed infinite sequence of distinct rational numbers

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0$$

where $P^{(k)} \neq 0$ and $Q^{(k)} \neq 0$ are integers such that

$$(P^{(k)}, Q^{(k)}) = 1.$$