PART 1

p-ADIC AND g-ADIC NUMBERS, AND THEIR APPROXIMATIONS

The theory of Diophantine approximations arose from that of indeterminate or Diophantine equations. This accounts for the name; no work by Diophant on Diophantine approximations is known.

A characteristic problem of the theory demands finding fractions $\frac{p}{q}$ of small denominators $q > 0$ that are near to a given real number α. One can then show in many ways that, if α is irrational, there are infinitely many such fractions satisfying

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2},$$

but that this is false if α is rational. In the opposite directions one can show, on one hand, that there are numbers α such that always

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}$$

where $c > 0$ is a constant, and, on the other hand, that there also exist numbers α for which the inequality

$$0 < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\omega}$$

has infinitely many solutions however large the exponent $\omega > 0$ is taken. We shall later study similar problems, but from a more general standpoint.

It seems probable that already the Greek mathematicians, in particular Archimedes, had methods for solving the last problem. For Euclid's algorithm for finding the greatest common divisor of two integers is closely connected with continued fractions, and such continued fractions give perhaps the simplest solution.

The general theory of Diophantine approximations may be said to have started with Lejeune-Dirichlet who introduced the *Schubfach-Prinzip* as a very powerful tool, particularly for the study of linear problems. More recently Minkowski founded the Geometry of Numbers which leads to even better results[1]). Other distinguished contributors to the theory were Jacobi, Liouville, Hermite, Kronecker, Thue, and Weyl, as well as several living mathematicians.

It is best not to restrict the theory to the study of real and complex numbers, but to allow other kinds of quantities and in particular the p-adic and g-adic numbers of K. Hensel. As the reader may not have studied such numbers previously I shall begin this course with their construction by means of valuation theory.

It will be assumed that the reader is acquainted with the elements of modern algebra, say with as much as is contained in the first five chapters of the well-known book by van der Waerden. I shall also assume a slight

1. We shall not apply the geometry of numbers; it is to form the subject and method of a continuation of these lectures.