ON AHLFORS'S THEORY OF COVERING SURFACES

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1. Introduction. In [1] (see [3, pp. 214-251]) Ahlfors introduced his theory of covering surfaces. His approach was combinatorial and geometric, and showed that R. Nevanlinna's theory of meromorphic functions had topological significance, and held in differentiated form. Other accounts are in [2], [5], [9], and [12] presents a very efficient proof using Ahlfors's own framework. See [10] for an independent approach, where the conclusions are slightly weaker than in [1].

Some years ago, John Lewis asked me if there was a way to derive Nevanlinna's value distribution theory directly from the argument principle. Since Nevanlinna's approach is based on Jensen's formula, itself the integrated argument principle, it is clear that the argument principle lies behind the theory, but the connection is, to say the least, highly indirect.

In this paper we show a more transparent connection. Very little is used that is not in a first course in complex analysis, but the subtleties needed to achieve (1.5) and (1.6) show the depth of Ahlfors's own insights. In retrospect our methods have considerable intersection with those of [1], although the orientation is different. I thank H. Donnelly, A. Eremenko, D. Gottlieb, L. Lempert, M. Ramachandran and A. Weitsman for helpful discussions. The idea for the latter part of Proposition (1.8) was shown to me by S. Lalley. The influence of Miles's work [8] is also apparent; see (2.23) below.

(1.1) Preliminaries. (See [1], [9, Ch. 13].) Let $a_1, \ldots, a_q$ be distinct (finite) complex numbers. We develop two situations in parallel: the "base surface" F_0 is either the Riemann sphere S^2 or $S^2 \setminus \bigcup_{k=1}^{q} D_k$, where the D_k are disjoint continuua about the a_k; we also let $a_{q+1} = \infty$ and take D_{q+1} accordingly. Thus, F_0 is either closed or bordered.

We impose a unit mass $\lambda(w)$ on F_0 with the properties specified in [1, I.1], [9, p. 325]; this allows lengths to be assigned to (Ahlfors)