Let $T \subseteq S$ and $S' = T \cup (S - T)$. Then S and S' are congruent.

Proof. First we have $S' \subseteq S$ because $T \subseteq S$ and $S - T \subseteq S$. Hence $\neg(Ex)(x \epsilon S \;\&\; x \notin S')$. Therefore it remains only to prove that $\neg(Ex)(x \notin S \;\&\; x \epsilon S')$. But this is equivalent to $\neg(Ex)(x \notin S \;\&\; (x \epsilon T \cdot v \cdot x \epsilon S \;\&\; x \notin T))$ which again is equivalent to $\neg(Ex)((x \notin S \;\&\; x \epsilon T) \; v \; (x \notin S \;\&\; x \epsilon S \;\&\; x \epsilon T))$ which is equivalent to $(Ex)(x \notin S \;\&\; x \epsilon T)$ which follows from $(x)(x \epsilon T \rightarrow x \epsilon S)$.

Simple examples of detachable subspecies of the natural number sequence are given by the even or the odd numbers. The linear continuum can be shown to have no other detachable subspecies than itself and the null species.

A species is said to be finite if there is a 1-to-1 correspondence between it and an initial part 1,..., n of the natural number series. It is called denumerable if there is such a correspondence between the species and the whole number series. A species is called numerable if it can be mapped onto a detachable subspecies of the sequence of natural numbers.

An important notion is "finitary spread" or, more briefly, "fan". A fan is a spread with such a spread law that there are only finitely many allowed first terms, and for every n every admitted sequence with n terms has only a finite number of sequences with n + 1 terms as admitted continuations. Above all the so-called fan theorem is important here. It says that if $\phi(\sigma)$ is an integral-valued function of σ, σ varying through the different elements of the fan, then the value of ϕ is already determined by a finite initial sequence of σ. Therefore, if $\phi(\sigma_1) = m$, there exists an n such that $\phi(\sigma_2) = m$ as often as σ_2 has the same first n terms as σ_1. An important application of the fan theorem is the proof of the statement that every function which is continuous on a bounded and closed point species is uniformly continuous on the point species. Further, such covering theorems as that of Heine-Borel can be proved. However, not all of the theorems of classical analysis can be proved in intuitionist mathematics.

I must confine my exposition of intuitionism to these scattered remarks A more thorough exposition would require a more complete treatment of intuitionist logic, and that would take more space than I have at my disposal here.

17. Mathematics without quantifiers

In all the theories we have treated above we have made use of the logical quantifiers, the universal one and the existential one. We have used them without scruples even in the case of an infinite number of objects. There is now a way of developing mathematics, in particular arithmetic, without the use of these operations which, in the case of an infinite number of objects, may be considered as an extension or extrapolation of conjunction and disjunction in the finite case. If we shall really consider the infinite as something becoming, something not finished or finishable, one might argue that we ought to avoid the quantifiers extended over an infinite range. Such a theory is possible. I myself published in 1923 a first beginning of such a strict finitist mathematics. I treated arithmetic, showing that by the use of