11. Some remarks on the nature of the set-theoretic axioms. The set-theoretic relativism.

Most of the axioms of the Zermelo-Fraenkel theory have the form: The class of all elements for which a certain statement is valid is a set, or, in other words, the domain D contains an element M such that all the objects in the class, and only these, are ϵ M. We might call these axioms "defining axioms," because the set which is declared to exist is also defined. There are two axioms at least, however, which are not of this kind, namely, the axiom of infinity and the axiom of choice. The axiom I mentioned expressing the general aleph hypothesis is of course not a defining axiom. As I have shown (see Mathematica Scandinavica, vol. 5, p. 40) the axiom of infinity can be put into defining form. The easiest way of doing that is to use the notion of ordinal set introduced in § 8. We may define a finite ordinal as an ordinal set M such that $(Ex)\ (x \epsilon M)\ \&\ (M = x^*)\ \&\ (y)(y \epsilon M \rightarrow (Ez)(z \epsilon y\ \&\ y = z^*)$. Here x^* means $x \cup \{x\}$. Then the axiom of infinity can be expressed by saying that the finite ordinals constitute a set.

The axiom of choice has given rise to many discussions. The reason for this is of course its non-constructive character. But people who desire to retain as much as possible of the old Cantor theory feel obliged to maintain that axiom. It is also quite clear that from an axiomatic point of view one must be allowed to study the consequences of any axioms whatever. On the other hand it cannot be denied that this axiom also leads to consequences which one scarcely had expected. I shall mention a couple of examples of this without entering into the proofs.

In Hausdorff's book "Grundzüge der Mengenlehre" one finds the proof of the following statement: It is possible to divide the surface of a sphere into 4 disjoint parts A,B,C,D such that A is a denumerable set of points, while B,C,D, are mutually congruent and at the same time B is congruent to C + D. That two sets of points are congruent means of course that they arise from one another by a rotation of the sphere.

Still more astonishing is a result obtained by Banach and Tarski which has later been improved by some other authors. In an article "Decompositions of a sphere" by T. J. Dekker and J. de Groot in Fund. Math. XLIII it is proved that it is possible to divide a 3-dimensional unit sphere in 5 disjoint pieces, each piece being a connected set, such that by suitable translations and rotations these pieces can be put together again so that two unit spheres are formed.

In the last instance it is a matter of personal taste whether one wants to have a set theory without or with an axiom of choice. A similar remark must be made with regard to the aleph hypothesis or the hypothesis of the existence of inaccessible cardinals etc.

From a purely logical point of view it would already be interesting to study a set theory with only defining axioms. I have proved (see my address "Some remarks on set theory" in the report of the International Congress of Mathematicians, Cambridge, Mass, 1950) that in such a theory the introduction of any set M can be brought into the form

$$(1) \qquad x \epsilon M \leftrightarrow \phi(x),$$