$x \leqq m_1$ is inductive finite. But according to a previous theorem then also the set of the $x \leqq m$ must be inductive finite. Therefore the set of all $x \leqq y$ is inductive finite for arbitrary y. Taking y then as the last element, one sees the truth of the theorem.

Using the last theorems we obtain another version of the proof of the statement that every inductive infinite set M is Dedekind infinite. However we must also use the well-ordering theorem, so that this proof depends on the axiom of choice as well. Let M be well-ordered. Then after our preceding results this well-ordering of M cannot simultaneously be an inverse well-ordering. Thus there is a subset $M_1 \supset 0$ without a last element. The set of all elements $x \leqq$ an element y of M_1 is then an initial part N of M without last element. Every element n of N has a successor $n' \epsilon N$. We may then define a mapping f of M into a proper part of M by putting $f(n) = n'$ for every $n \epsilon N$ and $f(n) = n$ for every n not ϵN.

10. The simple infinite sequence. Development of arithmetic

Let M be a Dedekind infinite set, f a one-to-one correspondence between M and a proper part M' of M. Let 0 denote an element of M not in M'. I denote generally by a' the image f(a) of a, also by P', when $P \subseteq M$, the set of all $p' = f(p)$ when p runs through P. Let N be the intersection of all subsets X of M possessing the two properties

1) $0 \epsilon X$, 2) $(x)(x \epsilon X \rightarrow x' \epsilon X)$.

Then N is called a simple infinite sequence or the f-chain from 0. We may say that it is the natural number series. It is evident that N has the properties 1) and 2). Further we have the principle of induction: A set containing 0 and for every x in it also containing x' contains N.

Theorem 46. $(y)(y \epsilon N \rightarrow (Ex)(y = x') \ \& \ (x \epsilon N) \cdot v \cdot y = 0)$.

This means that any element of N is either 0 or the f-image of another element of N. The proof is easy: Let us assume that $n \epsilon N$ and $\neq 0$ and $\neq$ every x' when $x \epsilon N$. Then $N - \{n\}$ would still possess the properties 1) and 2), which is absurd.

In order to develop arithmetic it is above all necessary to define the two fundamental operations addition and multiplication. Usually these as well as any other arithmetical functions are introduced by the so-called recursive definitions. I shall show how we are able to use here the ordinary explicit definitions which can be formulated with the aid of the predicate calculus. I shall introduce addition and multiplication by defining the sets of ordered triples (x,y,z) such that $x + y = z$ resp. $xy = z$.

We may consider the sets X of triples (a,b,c), where a,b,c are ϵN, which have the two properties:

1) All triples of the form (a,0,a) are ϵX.

2) Whenever (a,b,c) is ϵX, (a,b',c') is ϵX.