3. Axiomatic set theory. Axioms of Zermelo and Fraenkel

The discovery of the antinomies made it clear that a revision of the principles of set theory was necessary. The attempt to improve set theory which is best known among mathematicians is the axiomatic theory first set forth by Zermelo. I shall expose his theory in a somewhat more precise form, replacing his vague notion "definite Aussage" (= definite statement) by the notion proposition or propositional function in the first order predicate calculus. We assume that we are dealing with a domain D of objects together with the membership relation ϵ, so that all propositions are built up from atomic propositions of the form $x \epsilon y$ by use of the logical connectives &, v, -, $\rightarrow$ (and, or, not, if - when) and the quantifiers (x), (Ex) (for all x, for some x). Then the following axioms are assumed valid. I write them both in logical symbols and in ordinary language.

1. Axiom of extensionality.
   If x and y have just the same elements, then x = y. In symbols

$$(z)(z \epsilon x \leftrightarrow z \epsilon y) \rightarrow (x = y)$$

   Here x = y has the usual meaning, so that

$$(x = y) \rightarrow (U(x) \leftrightarrow U(y)),$$

   where U is an arbitrary predicate. Hence we also have

$$(x = y) \rightarrow (z)(x \epsilon z \leftrightarrow y \epsilon z)$$

2. Axiom of the small sets.
   a) There exists a set without elements denoted by the symbol 0. Because of 1. there can be only one such set.

$$(E x) (y) (y \bar{\epsilon} x).$$

   b) For every object m in D there exits a set $\{m\}$ containing m, but only m, as element,

$$(x)(Ey)(x \epsilon y \;\&\; (z)(z \epsilon y \rightarrow (z = x)))$$

   c) For all m and n in D there exists a set $\{m, n\}$ containing m and n, but only these, as elements.

$$(x)(y)(Ez)(x \epsilon z \;\&\; y \epsilon z \;\&\; (u)(u \epsilon z \rightarrow (u = x) \vee (u = y))) .$$

   Of course b) might be omitted because it follows from c) by putting n = m.

3. Axiom of separation.
   Let C(x) be a propositional function with x as the only free variable, and m an arbitrary set. Then there exists a set consisting of all elements x of m having the property C(x).

$$(x)(Ey)(z)(z \epsilon y \leftrightarrow C(z) \;\&\; z \epsilon x)$$