Appendix C

APPROXIMATION THEOREMS OVER ALGEBRAIC NUMBER FIELDS

1. Particularly the second part of these lectures was concerned mainly with the approximation of (real, p-adic, g-adic, and g*-adic) numbers by *rational numbers*. In this appendix certain results on the approximation of algebraic numbers by the elements of a fixed algebraic number field, of finite degree over the rational field, will be discussed. Without proofs, we shall state generalisations of the former Approximation Theorems. Detailed proofs would be rather long and involved; but there is no reason to forsee any essential difficulties.

One important special case was already investigated by W. J. LeVeque (Topics in Number Theory, Reading, Mass., 1956, vol. 2, 121-160). He proved the following generalisation of Roth's theorem.

Let K *be an algebraic number field of finite degree over the rational field* Γ; *let* ξ *be a real or complex algebraic number not in* K; *and let* $\tau > 2$. *There are at most finitely many elements* κ *of* K *satisfying*

$$|\xi-\kappa| < H(\kappa)^{-\tau} .$$

Here $H(\kappa)$ denotes the *height of* κ, i.e. the maximum of the absolute values of the coefficients of the irreducible primitive equation for κ with rational integral coefficients. LeVeque's proof shows quite clearly which changes, as compared with Roth's proof in the rational case, have to made for the theory over K.

The reader should also consult the paper by C. J. Parry (Acta mathematica 83, 1950, 1-100). This paper established $\mathfrak{p}$-adic generalisations of Siegel's theorem on the approximation of algebraic numbers by other algebraic numbers. Due to the new method by Roth, many of Parry's results can now be improved. Unfortunately, there still are difficulties if one wishes to study the approximation of a given algebraic number by *all algebraic numbers of a bounded degree* that do not necessarily lie in a fixed algebraic number field of finite degree.

2. As was already mentioned in Chapter 1, the rational field Γ has, except for equivalence, only the following valuations,

$$w_0(a), \ |a|, \ |a|_p;$$

here p runs over all different primes 2, 3, 5,... . These valuations are connected by the *basic identity*

$$w_0(a) = |a| \prod_p |a|_p .$$

Further every pseudo-valuation of Γ is equivalent either to a valuation of Γ, or to a pseudo-valuation of one of the two special types