Appendix B

A THEOREM BY M. CUGIANI

Roth's Theorem suggests the following problem.

Let ξ be a real algebraic number. To find a function $\epsilon(Q) > 0$ of the integral variable Q, with the property

$$\lim_{Q\to\infty} \epsilon(Q) = 0,$$

such that there are at most finitely many distinct rational numbers $\frac{P}{Q}$ with positive denominator for which

$$\left|\frac{P}{Q} - \xi\right| < Q^{-2-\epsilon(Q)} .$$

Unfortunately, the method of Roth does not seem strong enough for solving this problem and finding such a function $\epsilon(Q)$.

A weaker result may, however, be obtained and was, in fact, recently found by Marco Cugiani[1]. It states:

Theorem of Cugiani: *Let ξ be a real algebraic number of degree* f; *let*

$$\epsilon(Q) = 9f\,(\log\log\log Q)^{-\frac{1}{2}};$$

and let $\frac{P^{(1)}}{Q^{(1)}}, \frac{P^{(2)}}{Q^{(2)}}, \frac{P^{(3)}}{Q^{(3)}}, \ldots,$ *where* $e^e < Q^{(1)} < Q^{(2)} < Q^{(3)} < \ldots$, *be an infinite sequence of reduced rational numbers satisfying*

$$\left|\frac{P^{(k)}}{Q^{(k)}} - \xi\right| < Q^{(k)-2-\epsilon(Q^{(k)})} \qquad (k = 1,2,3,\ldots) .$$

Then

$$\limsup_{k\to\infty} \frac{\log Q^{(k+1)}}{\log Q^{(k)}} = \infty .$$

This theorem is thus an improvement of that by Th. Schneider[2] which was mentioned already in the Introduction to Part 2.

In this appendix we shall sketch a proof of the following theorem which contains Cugiani's result as the special case $\lambda = \mu = 1$.

Theorem 1: *Denote by $\xi \neq 0$ a real algebraic number of degree* f; *by* $g' \geq 2$ *and* $g'' \geq 2$ *two integers that are relatively prime; by λ and μ two real numbers satisfying*

1. Collectanea Mathematica, N. 169, Milano 1958.
2. J. reine angew. Math. 175 (1936), 182-192.