Appendix A

ANOTHER PROOF OF A LEMMA BY SCHNEIDER

The lemma by Schneider proved in §3 of Chapter 6 may also be obtained by means of a different method. This method has the advantage of leading to a slightly stronger result. It is due to my former colleague, G.E.H. Reuter, now professor of mathematics at the University of Durham.

1. A special case of Taylor's formula with Lagrange's error term states that if $f(x)$ is four times differentiable in a neighbourhood of $x=0$, then

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{IV}(\xi)\frac{x^4}{4!}$$

where ξ is a number between 0 and x. Let us apply this formula to the function $f(x)=\log\cosh x$ for non-negative values of x. Then

$$f'(x) = \tanh x, \quad f''(x) = \cosh^{-2}x, \quad f'''(x) = -2\sinh x \cosh^{-3}x$$

and

$$f^{IV}(x) = 4\cosh^{-2}x - 6\cosh^{-4}x.$$

The fourth derivative assumes its maximum when $\cosh x = \sqrt{3}$, and so

$$f^{IV}(x) \le \frac{2}{3} \qquad \text{for all } x \ge 0.$$

It follows therefore that

$$\log\cosh x \le \frac{1}{2}x^2 + \frac{2}{3}\cdot\frac{x^4}{24},$$

and hence that

(1): $$\cosh x \le \exp\left(\frac{1}{2}x^2 + \frac{1}{36}x^4\right) \qquad \text{if } x \ge 0.$$

2. Let again $r_1, \dots, r_m$ be m positive integers; let further $s, \rho_1, \dots, \rho_m$ be $m+1$ positive numbers. We denote by N the number of sets of m integers $(i_1, \dots, i_m)$ satisfying the inequalities

(2): $$0 \le i_1 \le r_1, \dots, 0 \le i_m \le r_m, \quad \sum_{h=1}^{m} \frac{i_h}{\rho_h} \le \left(\frac{1}{2} - s\right) \sum_{h=1}^{m} \frac{r_h}{\rho_h},$$

or, what is the same, the number of such sets satisfying

(3): $$0 \le i_1 \le r_1, \dots, 0 \le i_m \le r_m, \quad \sum_{h=1}^{m} \frac{i_h}{\rho_h} \ge \left(\frac{1}{2} + s\right) \sum_{h=1}^{m} \frac{r_h}{\rho_h}.$$

That both systems (2) and (3) have the same number of integral solutions is obvious because the transformation

$$(i_1, \dots, i_m) \to (r_1 - i_1, \dots, r_m - i_m)$$

interchanges their solutions.