Chapter 8

THE SECOND APPROXIMATION THEOREM

1. The two forms of the theorem.

This chapter contains a generalisation of the First Approximation Theorem which has just been proved. We begin by introducing some notations that will be used.

If α is any real number, and β is any p-adic number, put

$$|\alpha|^* = \min(|\alpha|, 1), \qquad |\beta|_p^* = \min(|\beta|_p, 1),$$

so that always

$$0 \leq |\alpha|^* \leq 1, \qquad 0 \leq |\beta|_p^* \leq 1.$$

Denote by

$$p_1, p_2, \ldots, p_r;\ p_{r+1}, p_{r+2}, \ldots, p_{r+r'};\ p_{r+r'+1}, p_{r+r'+2}, \ldots, p_{r+r'+r''}$$

a fixed system of

$$r+r'+r'', \qquad =n \text{ say},$$

distinct primes. It is *not* excluded that one, two, or all three of the numbers r, r', and r'', are equal to zero.

Let further

$$\xi \neq 0,\ \xi_1 \neq 0, \ldots, \xi_r \neq 0$$

denote a real algebraic number, a p_1-adic algebraic number, etc., a p_r-adic algebraic number, respectively. These algebraic numbers need *not* satisfy the same irreducible algebraic equation with rational coefficients, and thus they may belong to different finite extensions of the rational field.

Next let

$$F(x),\ F_1(x), \ldots, F_r(x)$$

be r+1 polynomials with rational coefficients, which neither vanish at x=0 nor have multiple factors. It is *not* required that all these polynomials are distinct, that they are irreducible, or that they are non-constant.

As in previous chapters, let again $\Sigma = \{\kappa^{(1)}, \kappa^{(2)}, \kappa^{(3)}, \ldots\}$ be an infinite sequence of distinct rational numbers

$$\kappa^{(k)} = \frac{P^{(k)}}{Q^{(k)}} \neq 0, \text{ where } P^{(k)} \neq 0,\ Q^{(k)} \neq 0,\ (P^{(k)}, Q^{(k)}) = 1,\ H^{(k)} = \max(|P^{(k)}|, |Q^{(k)}|).$$

Finally, put

$$\Phi(\kappa^{(k)}) = |\kappa^{(k)} - \xi|^* \prod_{j=1}^{r} |\kappa^{(k)} - \xi_j|_{p_j}^* \cdot \prod_{j=r+1}^{r+r'} |P^{(k)}|_{p_j} \cdot \prod_{j=r+r'+1}^{r+r'+r''} |Q^{(k)}|_{p_j}$$