Chapter 6

THE APPROXIMATION POLYNOMIAL

1. The aim.

In the present chapter we shall construct a polynomial

$$A(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1} \dots \sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1}\dots x_m^{i_m} \neq 0$$

which has integral coefficients that are not "not large" and which vanishes to a "very high" order at the point

$$x_1 = \xi,\dots,x_m = \xi;$$

here ξ is a given algebraic number. The importance of this approximation polynomial will become clear in the next chapters.

The construction does not involve valuation theory, but it is convenient to admit finite extensions of the rational number field.

2. The powers of an algebraic number.

Here and further on,

$$F(x) = F_0x^f + F_1x^{f-1} + \dots + F_f$$

denotes a fixed polynomial with integral coefficients such that

$$f \geq 1, \quad F_0 \neq 0, \quad F_f \neq 0$$

and therefore $F(0) \neq 0$. We impose the additional condition that $F(x)$ *has no multiple factor,* hence that $F(x)$ *and its derivative* $F'(x)$ *are relatively prime.*

Let Ω be an arbitrary (abstract) extension field of the rational field Γ in which $F(x)$ splits into a product of linear factors

$$F(x) = F_0(x-\xi_1)\dots(x-\xi_f).$$

The f zeros

$$\xi = \xi_1,\dots,\xi_f$$

of $F(x)$ are thus all *distinct* and *different from zero.*

We use the abbreviation

$$c = 2\max(|F_0|,|F_1|,\dots,|F_f|)$$

so that $c \geq 2$ is an integer.

Lemma 1: *For every exponent* $l=0, 1, 2,\dots$ *there exist unique integers* $g_0^{(l)}, g_1^{(l)},\dots,g_{f-1}^{(l)}$ *such that*