Chapter 5

ROTH'S LEMMA

1. Introduction

Roth bases the proof of his theorem on a general property of polynomials which is to be proved in this chapter. This property is roughly as follows.

Let

$$A(x_1,\dots,x_m) = \sum_{i_1=0}^{r_1}\cdots\sum_{i_m=0}^{r_m} a_{i_1\dots i_m} x_1^{i_1}\dots x_m^{i_m} \not\equiv 0$$

be a polynomial in m variables, with integral coefficients which are not "too large" in absolute values. Assume that

$$\max\left(\frac{r_2}{r_1}, \frac{r_3}{r_2},\dots,\frac{r_m}{r_{m-1}}\right)$$

is a "very small" positive number. Further let

$$\kappa_1 = \frac{P_1}{Q_1}, \dots, \kappa_m = \frac{P_m}{Q_m}$$

be m rational numbers written in their simplified forms for which both the maxima

$$H_1 = \max(|P_1|, |Q_1|),\dots, H_m = \max(|P_m|, |Q_m|)$$

and the quotients

$$\frac{\log H_2}{\log H_1}, \frac{\log H_3}{\log H_2},\dots,\frac{\log H_m}{\log H_{m-1}}$$

are "very large". Then $A(x_1,\dots,x_m)$ cannot vanish to a "very high" order at $x_1=\kappa_1,\dots,x_m=\kappa_m$. (An exact formulation of Roth's Lemma will be given at the end of this chapter).

The main idea of the proof consists in an induction for m, the number of variables, the case $m=1$ being trivial. This induction uses a test for *linear independence of polynomials* in terms of the so-called *generalised Wronski determinants.*

2. Linear dependence and independence.

Let

$$f_\nu = f_\nu(x_1,\dots,x_m) \qquad (\nu = 1,2,\dots, n)$$

be n rational functions of m variables, with coefficients in a field K. The functions are said to be *linearly dependent* (or for short, *dependent*) over K if there are elements $c_1,\dots, c_n$ of K not all zero such that