PART 2

RATIONAL APPROXIMATIONS OF ALGEBRAIC NUMBERS

The problem and its history.

Let α be a real algebraic number of degree $n \geq 2$; thus α is irrational. One of the results obtained in the proof of Theorem 1 of Chapter 3 was as follows. Let

$$F(x) = A_0x^m + A_1x^{m-1} + \ldots + A_m \neq 0$$

be any polynomial with integral coefficients, of degree at most m, and of height

$$A = \overline{|F(x)|} = \max(|A_0|, |A_1|, \ldots, |A_m|) \geq 1.$$

Then

$$\text{either} \quad F(\alpha) = 0 \quad \text{or} \quad |F(\alpha)| \geq c_1(m)A^{-(m-1)},$$

where $c_1(m) > 0$ depends on α and on m, but not on A.

Let now m=1 and $F(x) = Qx - P$ where $Q > 0$ and P are integers; then $A = \max(|P|, Q)$, and on putting $c_1 = c_1(1)$, the last result implies that

$$|Q\alpha - P| \geq c_1 \max(|P|, Q)^{-(n-1)},$$

because $Q\alpha - P \neq 0$. This inequality is equivalent to

$$(1): \qquad \left|\alpha - \frac{P}{Q}\right| \geq cQ^{-n}$$

where $c > 0$ is another constant depending only on α. For either

$$\left|\frac{P}{Q}\right| > |\alpha| + 1 \quad \text{and then} \quad \left|\alpha - \frac{P}{Q}\right| > 1 \geq Q^{-n},$$

or

$$\left|\frac{P}{Q}\right| \leq |\alpha| + 1, \text{ hence } \max(|P|, Q) \leq (|\alpha| + 1)Q, \text{ and then}$$

$$\left|\alpha - \frac{P}{Q}\right| \geq \frac{c_1}{Q}\{(|\alpha| + 1)Q\}^{-(n-1)} = \frac{c_1}{(|\alpha|+1)^{n-1}} Q^{-n}.$$

The inequality (1) is due to J. Liouville[1] who used it in his construction of real transcendental numbers. Apart from the value of the constant c, it is best possible for quadratic irrationals (n=2). For, as was proved in two different ways in Chapters 3 and 4, if α is any irrational number (not necessarily algebraic), then there are infinitely many distinct rational numbers $\frac{P}{Q}$ such that

1. C. R. Acad. Sci. (Paris), 18 (1844), 883-885, 910-911.