# Chapter 4

# CONTINUED FRACTIONS

Theorem 4 of the last chapter shows, in the special case when m=1, that there exist pairs of rational integers P, Q not both zero for which $\omega(Q\alpha - P)$ is arbitrarily small. The proof of the theorem, and that of Theorem 3 on which Theorem 4 is based, actually allows the effective construction of such approximations. However, the many pairs P, Q that have to be considered in order to find one satisfactory pair make this method prohibitive on account of the amount of labour that is required.

Free from this defect is a method based on the algorithm of continued fractions which is to be discussed in the present chapter.

The theory of continued fractions may go back to the times of classical Greek mathematics, as far as the real case is concerned; that for p-adic, g-adic, and g*-adic numbers, on the other hand, seems to be new in the form given here. Implicitly, they occur, however, already in an old paper of mine[1].

We shall begin with a short treatment of the regular continued fractions for real numbers. For further details the reader is referred to the standard works on the subject, e.g. to that by O. Perron. The continued fractions for p-adic, g-adic, and g*-adic numbers are then derived by means of a simple idea that makes use of the series for such numbers studied in Chapter 2.

## 1. The continued fraction algorithm in the real case.

Let $\alpha_0$ be any real number. Put

$$a_0 = [\alpha_0], \text{ so that } a_0 \leq \alpha_0 < a_0 + 1.$$

If $\alpha_0$ is an integer and hence $\alpha_0 = a_0$, this ends the algorithm. Otherwise put

$$\alpha_0 = a_0 + \frac{1}{\alpha_1}, \text{ where evidently } \alpha_1 > 1.$$

Put again

$$a_1 = [\alpha_1], \text{ so that } a_1 \leq \alpha_1 < a_1 + 1.$$

If now $\alpha_1$ is an integer, then $\alpha_1 = a_1$, and the algorithm again breaks off. In this manner we can continue. Either the algorith finally ends when we reach a number $\alpha_n$ which is an integer; or this never happens, and then the algorithm may be continued indefinitely. After n steps, it consists of the formulae,

$$\alpha_0 = a_0 + \frac{1}{\alpha_1}, \text{ where } a_0 = [\alpha_0], \alpha_1 > 1,$$

[1] Zur Approximation algebraischer Zahlen III, (1934), Acta math. 62, 91-166. See, in particular, the first part of this paper.