# Chapter 3

# A TEST FOR ALGEBRAIC OR TRANSCENDENTAL NUMBERS

Suppose a field K has been extended to its completion $K_w$ with respect to some valuation or pseudo-valuation w(a). The element $\alpha$ of $K_w$ is then said to be *algebraic* over K if it satisfies an equation

$$a_0x^n + a_1x^{n-1} + \dots + a_n = 0 \qquad (a_0 \neq 0,\ n \geqslant 1)$$

with coefficients in K, and it is otherwise called *transcendental* over K. The corresponding extension field $K(\alpha)$ is likewise algebraic, or transcendental, respectively.

Much of the following investigations are concerned with the problem whether a given real, p-adic, g-adic, or g*-adic number $\alpha$ is algebraic or transcendental over the rational field $\Gamma$. There is one, relatively elementary, approach to this problem where one studies the values of variable polynomials F(x) with rational integral coefficients at the point $x = \alpha$; it will be studied in the present chapter. A much deeper method, due to Thue, Siegel, and Roth, uses properties of the *rational approximations* of $\alpha$. For the explicit construction of such approximations a simple algorithm will be given in the next chapter. It is based on the continued fraction algorithm for real numbers and forms its natural extension to p-adic, g-adic, and g*-adic numbers. The deep theorem of Roth shows that these approximations cannot be too good in the case of an algebraic number $\alpha$. The theorem is best-possible and has many interesting consequences. The long and involved proof fills all the remaining chapters.

Before starting with these investigations, it has perhaps some interest to collect certain general properties that are basic for the theory of Diophantine approximations.

One such property was already mentioned in the first chapter. This was the

$$\textit{Fundamental Inequality:}\quad |a| \prod_{j=1}^{r} |a|_{p_j} \geqslant 1$$

where $a \neq 0$ is any rational integer, and $p_1, \dots, p_r$ are finitely many distinct primes. Thus, in particular,

$$|a| \geqslant 1 \quad \text{and} \quad |a|_p \geqslant \frac{1}{|a|}\ .$$

The second property makes a statement on the density of the rational integers on the real axis:

*If $\alpha$ and $\beta$ are real numbers such that $\alpha < \beta$, then there are exactly* $[\beta] - [\alpha]$ *rational integers* g *such that* $\alpha < g \leqslant \beta$. *In particular, the interval* $0 \leqslant g \leqslant \beta$ *contains exactly* $[\beta] + 1$, *and the interval* $-\beta \leqslant g \leqslant \beta$ *exactly* $2[\beta] + 1$ *rational integers.*