Chapter 2

THE p-ADIC, g-ADIC, AND g*-ADIC SERIES

Historically, K. Hensel was led to his p-adic and g-adic numbers by considerations of analogy to function fields.

Let Σ be the complex number field, x an indeterminate, and $K = \Sigma(x)$ a simple transcendental extension of Σ; let further $w(a)$ be any valuation or pseudo-valuation of K with the *property* C, i.e., such that

$$w(c) = w_0(c) \text{ if } c \in \Sigma,$$

where $w_0(a)$ denotes the trivial valuation defined in §1 of Chapter 1. It can be proved that every valuation with the property C must be equivalent to one of the valuations

$$w_0(a),\ ||a||,\ ||a||_{p_r}$$

introduced in § 3 of Chapter 1; however, now every "prime" p has the special form $p=x-c$ where $c\in\Sigma$ because Σ is algebraically closed. One can further show that every pseudo-valuation with the property C either is equivalent to one of these valuations, or it is equivalent to a pseudo-valuation of one of the two forms

$$w_1(a) = \max(||a||_{p_1},\ldots,||a||_{p_r}) \quad\text{and}\quad w_2(a) = \max(||a||,\ ||a||_{p_1},\ldots,||a||_{p_r}).$$

Here

$$p_1 = x - c_1,\ldots,\ p_r = x - c_r, \text{ where } c_h \neq c_k \text{ if } h \neq k,$$

are finitely many distinct "primes", and we have $r \geq 2$ in the case of $w_1(a)$ and $r \geq 1$ in that of $w_2(a)$. The position is thus analogous to that mentioned in § 14 of Chapter 1 for the rational field Γ, with $||a||$, $||a||_p$, $w_1(a)$, $w_2(a)$ corresponding to $|a|$, $|a|_p$, $|a|_g$, $|a|_{g^*}$, respectively, There is, however, the difference that all these valuations and pseudo-valuations of K are *Non-Archimedean.*

It is not difficult to prove that the completion of K with respect to $||a||$ is the field of all *formal* series

$$c_f\left(\frac{1}{x}\right)^f + c_{f+1}\left(\frac{1}{x}\right)^{f+1} + c_{f+2}\left(\frac{1}{x}\right)^{f+2} + \ldots$$

while that of K with respect to $||a||_p$, where $p=x-c$, is the field of all *formal* series

$$c_f(x-c)^f + c_{f+1}(x-c)^{f+1} + c_{f+2}(x-c)^{f+2} + \ldots .$$

In both cases f may be any rational integer, and the coefficients c_m may be arbitrary elements of Σ. The convergence of the series follows from the results in § 17 of Chapter 1 because

$$||\tfrac{1}{x}|| = \theta < 1,\ ||c_m|| \leq 1, \quad\text{and}\quad ||x-c||_p = \theta < 1,\ ||c_m||_p \leq 1,$$