Chapter 1

VALUATIONS AND PSEUDO-VALUATIONS

It is shown in abstract algebra that there are only the following two distinct types of simple extensions of a field K.

A *simple transcendental extension* of K is obtained by adjoining an indeterminate x to K and forming the field K(x) of all rational expressions in x with coefficients in K. Apart from isomorphisms there is only one such extension of K.

Next there are the *simple algebraic extensions* of K of which there may be many. These may be obtained as follows. Denote again by x an indeterminate, further by K[x] the ring of all polynomials in x with coefficients in K, and by f(x) an element of K[x] which is monic (i.e. has highest coefficient 1) and irreducible over K. The polynomials divisible by f(x) form a prime ideal $\mathfrak{p}$ in K[x]. Divide the elements of K[x] into residue classes modulo $\mathfrak{p}$ by putting two elements into the same class if their difference is in $\mathfrak{p}$. These residue classes form together the residue class ring $K[x]/\mathfrak{p}$ which, in fact, turns out to be a field. Furthermore, the residue class, ξ say, that contains the polynomial x, satisfies the equation $f(\xi) = 0$. In this way K has been extended to a field $K[x]/\mathfrak{p} = K(\xi)$ in which the equation $f(\xi) = 0$ has at least one root ξ. Apart from isomorphisms there is again only one such extension; but different monic irreducible polynomials f(x) will generate different simple algebraic extensions.

The construction of both extension fields K(x) and $K(\xi)$ does not require that K was already imbedded in a larger field, and it uses only algebraic processes. More important for the theory of Diophantine approximations is a non-algebraic method of field extension that is based on ideas from topology.

This non-algebraic method is applied already in elementary analysis where it serves to extend the field Γ of the rational numbers to the larger field P of the real numbers. Of the different variants of this method we select the one which has the advantage of easy generalization.

Define a real number α as the limit

$$\alpha = \lim_{m \to \infty} a_m$$

of a convergent sequence $\{a_m\} = \{a_1, a_2, a_3, \ldots\}$ of rational numbers; here the sequence is said to be convergent or a fundamental or Cauchy sequence if

$$\lim_{\substack{m \to \infty \\ n \to \infty}} |a_m - a_n| = 0.$$

Further two fundamental sequences $\{a_m\}$ and $\{b_m\}$ have the same limit if and only if

$$\lim_{m \to \infty} |a_m - b_m| = 0,$$

and the special sequence $\{a, a, a, \ldots\}$ has the rational limit a.