Preface

How many kinds of functions do you know? It is a typical topic in courses for university freshmen. The students first mention polynomial functions such as quadratic functions. Then they go on to name elementary functions such as fractional functions, algebraic functions expressed with square roots, a trigonometric function, the exponential function and the logarithmic function. Some precocious students might refer to elliptic functions, the Bessel functions, the zeta function. Some would even cite the Dirac delta function, and the professors respond by saying that the Dirac delta function is not a function.

In the beginning of the 20th century, what Paul Painlevé sought was to define new special functions. When we recall how elliptic functions enriched the mathematics of the 19th century, Painlevé's motivation seems quite natural.

Painlevé used differential equations to find new transcendental functions. By restricting to those satisfying differential equations, the zeta function in the above list is excluded. However, elliptic functions and the Bessel functions can be defined as the solution of simple differential equations.

Painlevé's research is a sequel to the studies on nonlinear ordinary differential equations in the complex domain by L. Fuchs, H. Poincaré and others. In general, the positions of the singularities for the solutions of nonlinear equations are not determined only by the equations; they depend on the initial conditions. Such a phenomenon is one of the difficulties in the analysis of nonlinear equations. We may observe this phenomenon just by considering the well-known differential equation satisfied by the elliptic function. The general solution of

$$\left(\frac{dy}{dt}\right)^2 = 4y^3 - g_2 y - g_3$$

can be expressed as $y = \wp(t-C)$, where $\wp$ is Weierstrass's elliptic function and C is an arbitrary constant. In this case, $t = C$ is a pole of the solution and its position is not determined only by the equation; it also depends on the initial condition.

Painlevé and Gambier classified the algebraic second order ordinary differential equations of normal form having solutions that admit at most poles as movable singularities. Such property is called the Painlevé property. Note that the singularities of the solutions of linear equations only come from the singularities of the coefficients of the equations, so that linear equations have the Painlevé property. In their classification of the algebraic second order ordinary differential equations with the Painlevé property, there were six types of equations whose solutions cannot be reduced to solutions of linear equations, elliptic functions or elementary functions. Nowadays, these six equations are called the Painlevé equations.

In 1970s, Wu, McCoy, Tracy, and Barouch discovered that the third Painlevé