As above we have

$$\| |u|^\lambda \|_{H^s((t_0,t_1)\times \mathbf{R}^n)} \le CX(u)^\lambda.$$

Applying Theorem 9.2.2, we obtain

$$\| |u(\sigma \cdot)|^\lambda \|_{H^{s,1/2}(X)} \le C \|u(\sigma \cdot)\|_{H^{s,1/2}(X)} \|u(\sigma \cdot)\|^{\lambda-1}_{L^\infty(X)}.$$

A comparison with (12.2.34) shows that

$$\Omega_0^{n/2} |u(\sigma\Omega)| \le \frac{C}{\sigma^{n/2}} X_{s,\rho}(u),$$

$$\sigma^{n/2} \|u(\sigma \cdot)\|_{H^{s,1/2}(X)} \le CX_{s,\rho}(u).$$

Hence, we arrive at

$$\int_{\rho_0}^{\rho} \| |u(\sigma \cdot)|^\lambda \|_{H^{s,1/2}(X)} \sigma^{n/2} d\sigma \le CX(u)^\lambda \int_{\rho_0}^{\rho} \sigma^{-(\lambda-1)n/2} d\sigma.$$

Our assumption that $\lambda > 1 + 2/n$ implies that

$$\int_{\rho_0}^{\rho} \sigma^{-(\lambda-1)n/2} d\sigma < \infty$$

so we have

$$\int_{\rho_0}^{\rho} \| |u(\sigma \cdot)|^\lambda \|_{H^{s,1/2}(X)} \sigma^{n/2} d\sigma \le CX(u)^\lambda.$$

This observation leads to the estimate

$$\rho^{n/2} \|N(u)(\rho \cdot)\|_{H^{s,1/2}(X)} \le C\varepsilon + X(u)^\lambda.$$

This estimate and (12.2.35) give (12.2.28) and completes the proof of (12.2.5).

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