Preface

Gauss hypergeometric functions and the functions in their family, such as Bessel functions, Whittaker functions, Hermite functions, Legendre polynomials and Jacobi polynomials etc. are the most fundamental and important special functions (cf. [**E–, Wa, WW**]). Many formulas related to the family have been studied and clarified together with the theory of ordinary differential equations, the theory of holomorphic functions and relations with other fields. They have been extensively used in various fields of mathematics, mathematical physics and engineering.

Euler studied the hypergeometric equation

$$x(1-x)y'' + \bigl(c-(a+b+1)x\bigr)y' - aby = 0 \tag{0.1}$$

with constant complex numbers a, b and c and he got the solution

$$F(a,b,c;x) := \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)\cdot b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)\cdot k!}x^k. \tag{0.2}$$

The series $F(a,b,c;x)$ is now called Gauss hypergeometric series or function and Gauss proved the Gauss summation formula

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{0.3}$$

when the real part of c is sufficiently large. Then in the study of this function an important concept was introduced by Riemann. That is the Riemann scheme

$$\begin{Bmatrix} x=0 & 1 & \infty & \\ 0 & 0 & a & ;\ x \\ 1-c & c-a-b & b & \end{Bmatrix} \tag{0.4}$$

which describes the property of singularities of the function and Riemann proved that this property characterizes the Gauss hypergeometric function.

The equation (0.1) is a second order Fuchsian differential equation on the Riemann sphere with the three singular points $\{0,1,\infty\}$. One of the main purpose of this paper is to generalize these results to the general Fuchsian differential equation on the Riemann sphere. In fact, our study will be applied to the following three kinds of generalizations.

One of the generalizations of the Gauss hypergeometric family is the hypergeometric family containing the generalized hypergeometric function ${}_nF_{n-1}(\alpha,\beta;x)$ or the solutions of Jordan-Pochhammer equations. Some of their global structures are concretely described as in the case of the Gauss hypergeometric family.

The second generalization is a class of Fuchsian differential equations such as the Heun equation which is of order 2 and has 4 singular points in the Riemann sphere. In this case, there appear *accessory parameters*. The global structure of the generic solution is quite transcendental and the Painlevé equation which describes the deformations preserving the monodromies of solutions of the equations with an apparent singular point is interesting and has been quite deeply studied and now it becomes an important field of mathematics.