Preface

The Riemann zeta function $\zeta(s)$ was first introduced and studied by L. Euler (1707–83) as a function of a real variable s greater than 1. On account of the behavior of $\zeta(s)$ as s approaches 1, he showed that the sum of the inverses of prime numbers is divergent (1737). After that, G.F.B. Riemann (1826–66) treated $\zeta(s)$ successfully as a function of a complex variable s and realized its importance in the study of the distribution of prime numbers (1859). For this reason, $\zeta(s)$ is called the Riemann zeta function. $\zeta(s)$ is holomorphic and has no zeros in the half-plane $\operatorname{Re} s$[†1] > 1. Riemann proved, moreover, that it has an analytic continuation to the whole complex plane, is meromorphic everywhere, and has a unique pole $s = 1$.

$\zeta(s)$ has no zeros in the closed half-plane $\operatorname{Re} s \geq 1$, and its zeros in the closed half-plane $\operatorname{Re} s \leq 0$ are only negative even integers $-2, -4, -6, \ldots$. Riemann conjectured that all zeros in the strip domain $0 < \operatorname{Re} s < 1$ lie on the line $\operatorname{Re} s = \frac{1}{2}$. This is called the Riemann hypothesis, which remains unsolved and is one of the Millennium Prize Problems posed by the Clay Mathematics Institute in 2000.

About the value-distribution of $\zeta(s)$ on the strip $\frac{1}{2} < \operatorname{Re} s \leq 1$, H. Bohr (1887–1951) showed that for fixed σ with $\frac{1}{2} < \sigma \leq 1$, the set of the values of $\zeta(\sigma + \sqrt{-1}t)$ where "time" t[†2] moves on the real line is dense in the complex plane (1914). Moreover he showed a similar result to hold for the log zeta function $\log \zeta$ (1915). Bohr was the first to develop the theory of almost periodic functions for his detailed study of the value-distribution of $\zeta(s)$ (1924). It was in the 1930s that he, together with B. Jessen, arrived at the final result:

> *There exists an asymptotic probability distribution of* $\log \zeta(\sigma + \sqrt{-1}\cdot)$*; roughly speaking, in a certain sense the time mean of the values of* $\log \zeta(\sigma + \sqrt{-1}t)$ *over a long finite time-interval is convergent to a probability distribution as the length of the time-interval goes to infinity.*

This statement is the Bohr-Jessen limit theorem, though it is restated in terms of the modern probability theory. After Bohr-Jessen, alternative proofs of the limit theorem were given by Jessen-Wintner [16], Borchsenius-Jessen [6], Laurinčikas [21, 22, 23], Matsumoto [24, 25] and others. They dealt with this theorem in the framework of probability theory, which originated with Jessen-Wintner and is a standard procedure nowadays.

[†1] For a complex number $s = \sigma + \sqrt{-1}t$, $\operatorname{Re} s$ denotes the real part σ and $\operatorname{Im} s$ the imaginary part t, where $\sqrt{-1}$ is the imaginary unit.

[†2] In this Preface only, but not in other places of this monograph, the variable t in $\zeta(\sigma + \sqrt{-1}t)$ as well as $\log \zeta(\sigma + \sqrt{-1}t)$ is regarded as and called "time" so as to be easy to understand what is said.