# Chapter 9

# APPLICATIONS

## 1. The theorems of Roth and Ridout.

The two Approximation Theorems proved in Chapters 7 and 8 contain as special cases the theorems of Roth and of Ridout. These results were already mentioned in the introduction to Part 2, and we then gave the references to the literature.

Roth's theorem may be considered as either the special case $d=1$, $\lambda=1$, $\mu=1$ of Theorem (1,I) or as the special case $r=r'=r''=0$ of Theorem (2,I). It states:

*If $\xi$ is a real algebraic number; if $\rho$ and $c_1$ are positive constants; and if there exist infinitely many distinct simplified fractions $\frac{P^{(k)}}{Q^{(k)}}$ such that*

$$\left|\frac{P^{(k)}}{Q^{(k)}} - \xi\right| \leq c_1 |Q^{(k)}|^{-\rho},$$

*then $\rho \leq 2$.*

This theorem is obvious if $\xi$ is rational; thus, e.g. the case when $\xi = 0$ may be excluded. Now $\frac{P^{(k)}}{Q^{(k)}}$ tends to $\xi$ as $k$ tends to infinity. Hence, by $\xi \neq 0$, the integers $P^{(k)}$, $Q^{(k)}$, and $H^{(k)}$ have the same order of magnitude. It follows that, for all $k$,

$$c_1 |Q^{(k)}|^{-\rho} \leq c_1' H^{(k)-\rho}$$

where $c_1'$ is a further positive constant. Hence the assertion is an immediate consequence of either Approximation Theorem.

We next show that Ridout's theorems may be deduced from Theorem (2,I).

His first theorem is as follows. Let again $\xi \neq 0$ be a real algebraic number; let $\rho$, $c_1$, $c_3$, and $c_4$ be positive constants; and let $\lambda$ and $\mu$ be constants such that

$$0 \leq \lambda \leq 1, \quad 0 \leq \mu \leq 1.$$

*Assume that there exist infinitely many simplified fractions $\frac{P^{(k)}}{Q^{(k)}}$ with the following properties:*

(i): $$\left|\frac{P^{(k)}}{Q^{(k)}} - \xi\right| \leq c_1 |Q^{(k)}|^{-\rho} .$$

(ii): *The numerators $P^{(k)}$ and the denominators $Q^{(k)}$ are distinct from zero*