Comments on Selected Problems

CHAPTER 1

4. This problem gives the direct sum version of partitioned matrices. For (ii), identify V_1 with vectors of the form $\{v_1, 0\} \in V_1 \oplus V_2$ and restrict T to these. This restriction is a map from V_1 to $V_1 \oplus V_2$ so $T\{v_1, 0\} = \{z_1(v_1), z_2(v_1)\}$ where $z_1(v_1) \in V_1$ and $z_2(v_1) \in V_2$. Show that z_1 is a linear transformation on V_1 to V_1 and z_2 is a linear transformation on V_1 to V_2. This gives A_{11} and A_{21}. A similar argument gives A_{12} and A_{22}. Part (iii) is a routine computation.
5. If $x_{r+1} = \Sigma_1^r c_i x_i$, then $w_{r+1} = \Sigma_1^r c_i w_i$.
8. If $u \in R^k$ has coordinates $u_1, \ldots, u_k$, then $Au = \Sigma_1^k u_i x_i$ and all such vectors are just span $\{x_1, \ldots, x_k\}$. For (ii), $r(A) = r(A')$ so $\dim \mathfrak{R}(A'A) = \dim \mathfrak{R}(AA')$.
10. The algorithm of projecting $x_2, \ldots, x_k$ onto $\{\text{span } x_1\}^\perp$ is known as Björk's algorithm (Björk, 1967) and is an alternative method of doing Gram–Schmidt. Once you see that $y_2, \ldots, y_k$ are perpendicular to y_1, this problem is not hard.
11. The assumptions and linearity imply that $[Ax, w] = [Bx, w]$ for all $x \in V$ and $w \in W$. Thus $[(A - B)x, w] = 0$ for all w. Choose $w = (A - B)x$ so $(A - B)x = 0$.
12. Choose z such that $[y_1, z] \neq 0$. Then $[y_1, z]x_1 = [y_2, z]x_2$ so set $c = [y_2, z]/[y_1, z]$. Thus $cx_2 \square y_1 = x_2 \square y_2$ so $cy_1 \square x_2 = y_2 \square x_2$. Hence $c\|x_2\|^2 y_1 = \|x_2\|^2 y_2$ so $y_1 = c^{-1} y_2$.
13. This problem shows the topologies generated by inner products are all the same. We know $[x, y] = (x, Ay)$ for some $A > 0$. Let c_1 be the minimum eigenvalue of A, and let c_2 be the maximum eigenvalue of A.