The Arithmetical Hierarchy Theorem shows that there are no inclusions among the classes Π_n^0 and Σ_n^0 other than those given by 13.2.

The Arithmetical Enumeration Theorem is false for Δ_n^0 relations; for if it were true, we could use the proof of the Arithmetical Hierarchy Theorem to show that there is a Δ_n^0 relation which is not Δ_n^0.

Let Φ be a set of total functions. If Q is any concept defined in terms of recursive functions, we can obtain a definition of Q in Φ or relative to Φ by replacing recursive everywhere in the definition of Q by recursive in Φ. For example, R is arithmetical in Φ if it has a definition (1) where P is recursive in Φ; and R is Π_n^0 in Φ if it has such a definition in which the prefix is Π_n^0. We shall assume that this is done for all past and future definitions.

Now let us consider how the results of this section extend to the relativized case. Up to the Enumeration Theorem, everything extends without problems. The rest extends to finite Φ but not to arbitrary Φ. For example, if Φ is the set of all reals, then every unary relation is recursive in Φ and hence Π_n^0 and Σ_n^0 in Φ for all n. Thus the Hierarchy Theorem fails. Since the Hierarchy Theorem is a consequence of the Enumeration Theorem, the Enumeration Theorem also fails.

14. Recursively Enumerable Relations

A relation R is semicomputable if there is an algorithm which, when applied to the inputs $\vec{x}$, gives an output iff $R(\vec{x})$. If F is the function computed by the algorithm, then the algorithm applied to $\vec{x}$ gives an output iff $\vec{x}$ is in the domain of F. Hence R is semicomputable iff it is the domain of a computable function.

As an example, let A be the set of n such that $x^n + y^n = z^n$ holds for some positive integers x, y, and z. Then A is semicomputable; the algorithm with input n tests each triple (x,y,z) in turn to see if $x^n + y^n = z^n$. On the other