ON ω_1-COMPLETE FILTERS

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Let us start with a definition. For an uncountable cardinal κ set

$$\mu(\kappa) = \min\{|H| \mid H \text{ is a set of } \omega_1\text{-complete uniform filters on } \kappa \text{ and } \forall A \subseteq \kappa\, \exists F \in H (A \in F \text{ or } \kappa - A \in F)\}$$

Clearly, $1 \leq \mu(\kappa) \leq 2^\kappa$.

A classical result of Ulam says that κ must be very large, if $\mu(\kappa) = 1$. On the other hand, by definition we have that $\mu(\kappa) = 1$ if κ is bigger than some strongly compact cardinal. Only recently (see [3]) Gitik has shown that $\mu(\kappa) \leq \omega$ implies that $\mu(\kappa) = 1$.

Can $\mu(\kappa)$ be small for small cardinals κ? Using a huge cardinal, Magidor showed in [4] that $\mu(\omega_3) \leq \omega_3$ is consistent. Shelah constructed a model of $\mu(\omega_1) = \omega_1$ starting with many supercompact cardinals (see [6]). With an almost huge cardinal Woodin produced a model where $\mu(\omega_1) = \omega_1$ is witnessed by normal filters. It seems to be an open problem whether $\mu(\omega_2) = \omega_1$ is consistent.

In this note we treat the following question. Is there always some κ such that $\mu(\kappa) \leq \kappa$? Prikry showed in [5] that $\mu(\omega_1) > \omega_1$ is consistent. Jensen showed later that the appropriate combinatorial principle holds in L which implies that $\mu(\omega_1) > \omega_1$ is true in L. We shall show:

THEOREM 1. *Assume $V = L$. Then $\mu(\kappa) > \kappa$ for all regular $\kappa > \omega$.*

To prove this we reduce the problem to a purely combinatorial question. So let us introduce the following principle. Let $\kappa > \omega$ be regular. Then Q_κ denotes the following property:

> There is some $G \subseteq \{f \mid f : \kappa \to 2\}$ such that $|G| > \kappa$ and for all $G^* \subseteq G$ such that $|G^*| > \kappa$ there is a countable $\overline{G} \subseteq G^*$ such that $\{\alpha < \kappa \mid \forall f, g \in \overline{G}, f(\alpha) = g(\alpha)\}$ is nonstationary.

This principle is closely related to some properties discussed in [7]. So the interested reader might also consult that paper. Now we have:

LEMMA 1. *Let $\kappa > \omega$ be regular and assume that Q_κ holds. Then $\mu(\kappa) > \kappa$.*

Proof. Assume not. Let $\mu(\kappa) \leq \kappa$ be given by H. By a result of Taylor (see [8]) we may assume that all $F \in H$ contain the club filter on κ. Let Q_κ be given by G. For each $f \in G$ choose $F_f \in H$ and $i_f < 2$ such that $\{\alpha < \kappa \mid f(\alpha) = i_f\} \in F_f$. Then there are $G^* \subseteq G$, $i < 2$, $F \in H$ such that