31 Borel metric spaces and lines in the plane

We give two applications of Harrington's technique of using Gandy forcing. First let us begin by isolating a principal which we call overflow. It is an easy consequence of the Separation Theorem.

Lemma 31.1 *(Overflow) Suppose $\theta(x_1, x_2, \ldots, x_n)$ is a Π^1_1 formula and A is a Σ^1_1 set such that*

$$\forall x_1, \ldots, x_n \in A \;\; \theta(x_1, \ldots, x_n).$$

Then there exists a Δ^1_1 set $D \supseteq A$ such that

$$\forall x_1, \ldots, x_n \in D \;\; \theta(x_1, \ldots, x_n).$$

proof:

For $n = 1$ this is just the Separation Theorem 27.5.

For $n = 2$ define

$$B = \{x : \forall y(y \in A \rightarrow \theta(x, y))\}.$$

Then B is Π^1_1 set which contains A. Hence by separation there exists a Δ^1_1 set E with $A \subseteq E \subseteq B$. Now define

$$C = \{x : \forall y(y \in E \rightarrow \theta(x, y)\}.$$

Then C is a Π^1_1 set which also contains A. By applying separation again we get a Δ^1_1 set F with $A \subseteq F \subseteq C$. Letting $D = E \cap F$ does the job. The proof for $n > 2$ is similar.

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We say that (B, δ) is a *Borel metric space* iff B is Borel, δ is a metric on B, and for every $\epsilon \in \mathbb{Q}$ the set

$$\{(x, y) \in B^2 : \delta(x, y) \leq \epsilon\}$$

is Borel.

Theorem 31.2 *(Harrington [39]) If (B, δ) is a Borel metric space, then either (B, δ) is separable (i.e., contains a countable dense set) or for some $\epsilon > 0$ there exists a perfect set $P \subseteq B$ such that $\delta(x, y) > \epsilon$ for every distinct $x, y \in P$.*

proof:

By relativizing the proof to an arbitrary parameter we may assume that B and the sets $\{(x, y) \in B^2 : \delta(x, y) \leq \epsilon\}$ are Δ^1_1.

Lemma 31.3 *For any $\epsilon \in \mathbb{Q}^+$ if $A \subseteq B$ is Σ^1_1 and the diameter of A is less than ϵ, then there exists a Δ^1_1 set D with diameter less than ϵ and $A \subseteq D \subseteq B$.*