Part IV
Gandy Forcing

30 Π^1_1 equivalence relations

Theorem 30.1 *(Silver [99]) Suppose (X,E) is a $\underset{\sim}{\Pi}^1_1$ equivalence relation, i.e. X is a Borel set and $E \subseteq X^2$ is a $\underset{\sim}{\Pi}^1_1$ equivalence relation on X. Then either E has countably many equivalence classes or there exists a perfect set of pairwise inequivalent elements.*

Before giving the proof consider the following example. Let WO be the set of all characteristic functions of well-orderings of ω. This is a Π^1_1 subset of $2^{\omega\times\omega}$. Now define $x \simeq y$ iff there exists an isomorphism taking x to y or $x, y \notin WO$. Note that $(2^{\omega\times\omega}, \simeq)$ is a Σ^1_1 equivalence relation with exactly ω_1 equivalence classes. Furthermore, if we restrict $\simeq$ to WO, then $(WO, \simeq)$ is a Π^1_1 equivalence relation (since well-orderings are isomorphic iff neither is isomorphic to an initial segment of the other). Consequently, Silver's theorem is the best possible.

The proof we are going to give is due to Harrington [33], see also Kechris and Martin [51], Mansfield and Weitkamp [71] and Louveau [62]. A model theoretic proof is given in Harrington and Shelah [38].

We can assume that X is Δ^1_1 and E is Π^1_1, since the proof readily relativizes to an arbitrary parameter. Also, without loss, we may assume that $X = \omega^\omega$ since we just make the complement of X into one more equivalence class.

Let $\mathbb{P}$ be the partial order of nonempty Σ^1_1 subsets of ω^ω ordered by inclusion. This is known as *Gandy forcing.* Note that there are many trivial generic filters corresponding to Σ^1_1 singletons.

Lemma 30.2 *If G is $\mathbb{P}$-generic over V, then there exists $a \in \omega^\omega$ such that $G = \{p \in \mathbb{P} : a \in p\}$ and $\{a\} = \bigcap G$.*

proof:

For every n an easy density argument shows that there exists a unique $s \in \omega^n$ such that $[s] \in G$ where $[s] = \{x \in \omega^\omega : s \subseteq x\}$. Define $a \in \omega^\omega$ by $[a \upharpoonright n] \in G$ for each n. Clearly, $\bigcap G \subseteq \{a\}$.

Now suppose $B \in G$, we need to show $a \in B$. Let $B = p[T]$.

Claim: There exists $x \in \omega^\omega$ such that $p[T^{x\upharpoonright n, a\upharpoonright n}] \in G$ for every $n \in \omega$.

proof:

This is by induction on n. Suppose $p[T^{x\upharpoonright n, a\upharpoonright n}] \in G$. Then

$$p[T^{x\upharpoonright n, a\upharpoonright n+1}] \in G$$

since

$$p[T^{x\upharpoonright n, a\upharpoonright n+1}] = [a \upharpoonright n+1] \cap p[T^{x\upharpoonright n, a\upharpoonright n}]$$