29 Δ^1_1-codes

Using Π^1_1-reduction and universal sets it is possible to get codes for Δ^1_1 subsets of ω and ω^ω.

Here is what we mean by Δ^1_1 codes for subsets of X where $X = \omega$ or $X = \omega^\omega$.

There exists a Π^1_1 sets $C \subseteq \omega \times \omega^\omega$ and $P \subseteq \omega \times \omega^\omega \times X$ and a Σ^1_1 set $S \subseteq \omega \times \omega^\omega \times X$ such that

- for any $(e, u) \in C$
$$\{x \in X : (e, u, x) \in P\} = \{x \in X : (e, u, x) \in S\}$$
- for any $u \in \omega^\omega$ and $\Delta^1_1(u)$ set $D \subseteq X$ there exists a $(e, u) \in C$ such that
$$D = \{x \in X : (e, u, x) \in P\} = \{x \in X : (e, u, x) \in S\}.$$

From now on we will write
"e is a $\Delta^1_1(u)$-code for a subset of X"
to mean $(e, u) \in C$ and remember that it is a Π^1_1 predicate.

We also write "D is the $\Delta^1_1(u)$ set coded by e" if "e is a $\Delta^1_1(u)$-code for a subset of X" and

$$D = \{x \in X : (e, x) \in P\} = \{x \in X : (e, x) \in S\}.$$

Note that $x \in D$ can be said in either a $\Sigma^1_1(u)$ way or $\Pi^1_1(u)$ way, using either S or P.

Theorem 29.1 *(Spector-Gandy [103][31]) Π^1_1-reduction and universal sets implies Δ^1_1 codes exist.*

proof:

Let $U \subseteq \omega \times \omega^\omega \times X$ be a Π^1_1 set which is universal for all $\Pi^1_1(u)$ sets, i.e., for every $u \in \omega^\omega$ and $A \in \Pi^1_1(u)$ with $A \subseteq X$ there exists $e \in \omega$ such that $A = \{x \in X : (e, u, x) \in U\}$. For example, to get such a U proceed as follows. Let $\{e\}^u$ be the partial function you get by using the e^{th} Turing machine with oracle u. Then define $(e, u, x) \in U$ iff $\{e\}^u$ is the characteristic function of a tree $T \subseteq \bigcup_{n<\omega}(\omega^n \times \omega^n)$ and $T_x = \{s : (s, x \upharpoonright |s|) \in T\}$ is well-founded.

Now get a doubly universal pair. Let $e \mapsto (e_0, e_1)$ be the usual recursive unpairing function from ω to $\omega \times \omega$ and define

$$U^0 = \{(e, u, x) : (e_0, u, x) \in U\}$$

and

$$U^1 = \{(e, u, x) : (e_1, u, x) \in U\}.$$

The pair of sets U^0 and U^1 are Π^1_1 and doubly universal, i.e., for any $u \in \omega^\omega$ and A and B which are $\Pi^1_1(u)$ subsets of X there exists $e \in \omega$ such that

$$A = \{x : (e, u, x) \in U^0\}$$