27 Kleene Separation Theorem

We begin by defining the hyperarithmetic subsets of ω^ω. We continue with our view of Borel sets as well-founded trees with little dohickey's (basic clopen sets) attached to its terminal nodes.

A *code for a hyperarithmetic set* is a triple (T, p, q) where T is a recursive well-founded subtree of $\omega^{<\omega}$, $p : T^{>0} \to 2$ is recursive, and $q : T^0 \to \mathcal{B}$ is a recursive map, where $\mathcal{B}$ is the set of basic clopen subsets of ω^ω including the empty set. Given a code (T, p, q) we define $\langle C_s : s \in T\rangle$ as follows.

- if s is a terminal node of T, then
$$C_s = q(s)$$
- if s is a not a terminal node and $p(s) = 0$, then
$$C_s = \bigcup\{C_{s^\frown n} : s^\frown n \in T\},$$
  and
- if s is a not a terminal node and $p(s) = 1$, then
$$C_s = \bigcap\{C_{s^\frown n} : s^\frown n \in T\}.$$

Here we are being a little more flexible by allowing unions and intersections at various nodes.

Finally, the set C coded by (T, p, q) is the set $C_{\langle\rangle}$. A set $C \subseteq \omega^\omega$ is hyperarithmetic iff it is coded by some recursive (T, p, q).

Theorem 27.1 *(Kleene [53]) Suppose A and B are disjoint Σ^1_1 subsets of ω^ω. Then there exists a hyperarithmetic set C which separates them, i.e., $A \subseteq C$ and $C \cap B = \emptyset$.*

proof:

This amounts basically to a constructive proof of the classical Separation Theorem 26.1.

Let $A = p[T_A]$ and $B = p[T_B]$ where T_A and T_B are recursive subtrees of $\bigcup_{n\in\omega}(\omega^n \times \omega^n)$, and
$$p[T_A] = \{y : \exists x \forall n \quad (x \upharpoonright n, y \upharpoonright n) \in T_A\}$$
and similarly for $p[T_B]$. Now define the tree
$$T = \{(u, v, t) : (u, t) \in T_A \text{ and } (v, t) \in T_B\}.$$

Notice that T is recursive tree which is well-founded. Any infinite branch thru T would give a point in the intersection of A and B which would contradict the fact that they are disjoint.