25 Large Π^1_2 sets

A set is $\mathbf{\Pi}^1_2$ iff it is the complement of a $\mathbf{\Sigma}^1_2$ set. Unlike $\mathbf{\Sigma}^1_2$ sets which cannot have size strictly in between ω_1 and the continuum (Theorem 21.1), $\mathbf{\Pi}^1_2$ sets can be practically anything.[11]

Theorem 25.1 *(Harrington [35]) Suppose V is a model of set theory which satisfies $\omega_1 = \omega_1^L$ and B is arbitrary subset of ω^ω in V. Then there exists a ccc extension of V, $V[G]$, in which B is a $\mathbf{\Pi}^1_2$ set.*

proof:

Let $\mathbb{P}_B$ be the following poset. $p \in \mathbb{P}_B$ iff p is a finite consistent set of sentences of the form:

1. "$[s] \cap \overset{\circ}{C}_n = \emptyset$", or
2. "$x \in \overset{\circ}{C}_n$, where $x \in B$.

This partial order is isomorphic to Silver's view of almost disjoint sets forcing (Theorem 5.1). So forcing with $\mathbb{P}_B$ creates an F_σ set $\bigcup_{n\in\omega} C_n$ so that

$$\forall x \in \omega^\omega \cap V (x \in B \text{ iff } x \in \bigcup_{n<\omega} C_n).$$

Forcing with the direct sum of ω_1 copies of $\mathbb{P}_B$, $\prod_{\alpha<\omega_1} \mathbb{P}_B$, we have that

$$\forall x \in \omega^\omega \cap V[\langle G_\alpha : \alpha < \omega_1\rangle](x \in B \text{ iff } x \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C_n^\alpha).$$

One way to see this is as follows. Note that in any case

$$B \subseteq \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C_n^\alpha.$$

So it is the other implication which needs to be proved. By ccc, for any $x \in V[\langle G_\alpha : \alpha < \omega_1\rangle]$ there exists $\beta < \omega_1$ with $x \in V[\langle G_\alpha : \alpha < \beta\rangle]$. But considering $V[\langle G_\alpha : \alpha < \beta\rangle]$ as the new ground model, then G_β would be $\mathbb{P}_B$- generic over $V[\langle G_\alpha : \alpha < \beta\rangle]$ and hence if $x \notin B$ we would have $x \notin \cup_{n<\omega} C_n^\beta$.

Another argument will be given in the proof of the next lemma.

Lemma 25.2 *Suppose $\langle c_\alpha : \alpha < \omega_1$ be a sequence in V of elements of ω^ω and $\langle a_\alpha : \alpha < \omega_1\rangle$ is a sequence in $V[\langle G_\alpha : \alpha < \omega_1\rangle]$ of elements of 2^ω. Using Silver's forcing add a sequence of $\mathbf{\Pi}^0_2$ sets $\langle U_n : n < \omega\rangle$ such that*

$$\forall n \in \omega \forall \alpha < \omega_1 (a_\alpha(n) = 1 \textit{ iff } c_\alpha \in U_n).$$

Then

$$V[\langle G_\alpha : \alpha < \omega_1\rangle][\langle U_n : n < \omega\rangle] \models \forall x \in \omega^\omega \ (x \in B \textit{ iff } x \in \bigcap_{\alpha<\omega_1} \cup_{n<\omega} C_n^\alpha).$$

[11] It's life Jim, but not as we know it.- Spock of Vulcan