22 Uniformity and Scales

Given $R \subseteq X \times Y$ we say that $S \subseteq X \times Y$ uniformizes R iff

1. $S \subseteq R$,

2. for all $x \in X$ if there exists $y \in Y$ such that $R(x, y)$, then there exists $y \in Y$ such that $S(x, y)$, and

3. for all $x \in X$ and $y, z \in Y$ if $S(x, y)$ and $S(x, z)$, then $y = z$.

Another way to say the same thing is that S is a subset of R which is the graph of a function whose domain is the same as R's.

Theorem 22.1 *(Kondo [47]) Every Π^1_1 set R can be uniformized by a Π^1_1 set S.*

Here, X and Y can be taken to be either ω or ω^ω or even a singleton $\{0\}$. In this last case, this amounts to saying for any nonempty Π^1_1 set $A \subseteq \omega^\omega$ there exists a Π^1_1 set $B \subseteq A$ such that B is a singleton, i.e., $|B| = 1$. The proof of this Theorem is to use a property which has become known as the Scale property.

Lemma 22.2 *(Scale property) For any Π^1_1 set A there exists $\langle \phi_i : i < \omega \rangle$ such that*

1. *each $\phi_i : A \to \mathrm{OR}$,*

2. *for all i and $x, y \in A$ if $\phi_{i+1}(x) \leq \phi_{i+1}(y)$, then $\phi_i(x) \leq \phi_i(y)$,*

3. *for every $x, y \in A$ if $\forall i \;\; \phi_i(x) = \phi_i(y)$, then $x = y$,*

4. *for all $\langle x_n : n < \omega \rangle \in A^\omega$ and $\langle \alpha_i : i < \omega \rangle \in \mathrm{OR}^\omega$ if for every i and for all but finitely many n $\phi_i(x_n) = \alpha_i$, then there exists $x \in A$ such that $\lim_{n \to \infty} x_n = x$ and for each i $\phi_i(x) \leq \alpha_i$,*

5. *there exists P a Π^1_1 set such that for all $x, y \in A$ and i*

   $$P(i, x, y) \text{ iff } \phi_i(x) \leq \phi_i(y)$$

   and for all $x \in A$, $y \notin A$, $i \in \omega$ $P(i, x, y)$, and

6. *there exists S a Σ^1_1 set such that for all $x, y \in A$ and i*

   $$S(i, x, y) \text{ iff } \phi_i(x) \leq \phi_i(y)$$

   and for all $y \in A$, x, $i \in \omega$ if $S(i, x, y)$, then $x \in A$.

Another way to view a scale is from the point of view of the relations on A defined by $x \leq_i y$ iff $\phi_i(x) \leq \phi)i(y)$. These are called prewellorderings. They are well orderings if we mod out by $x \equiv_i y$ which is defined by

$$x \equiv_i y \text{ iff } x \leq_i y \text{ and } y \leq_i x.$$