21 Mansfield-Solovay Theorem

Theorem 21.1 *(Mansfield [70], Solovay [101]) If $A \subseteq \omega^\omega$ is a Σ^1_2 set with constructible parameter which contains a nonconstructible element of ω^ω, then A contains a perfect set which is coded in L.*

proof:

By Shoenfield's Theorem 20.1, we may assume $A = p[T]$ where $T \in L$ and $T \subseteq \bigcup_{n<\omega} \omega_1^n \times \omega^n$. Working in L define the following decreasing sequence of subtrees as follows.

$T_0 = T$,

$T_\lambda = \bigcap_{\beta<\lambda} T_\beta$, if λ a limit ordinal, and

$T_{\alpha+1} = \{(r,s) \in T_\alpha : \exists (r_0,s_0),(r_1,s_1) \in T_\alpha$ such that $(r_0,s_0),(r_1,s_1)$ extend (r,s), and s_0 and s_1 are incompatible$\}$.

Each T_α is tree, and for $\alpha < \beta$ we have $T_\beta \subseteq T_\alpha$. Thus there exists some α_0 such that $T_{\alpha_0+1} = T_{\alpha_0}$.

Claim: $[T_{\alpha_0}]$ is nonempty.

proof:

Let $(x,y) \in [T]$ be any pair with y not constructible. Since $A = p[T]$ and A is not a subset of L, such a pair must exist. Prove by induction on α that $(x,y) \in [T_\alpha]$. This is easy for α a limit ordinal. So suppose $(x,y) \in [T_\alpha]$ but $(x,y) \notin [T_{\alpha+1}]$. By the definition it must be that there exists $n < \omega$ such that $(x \restriction n, y \restriction n) = (r,s) \notin T_{\alpha+1}$. But in L we can define the tree:

$$T_\alpha^{(r,s)} = \{(\hat{r},\hat{s}) \in T_\alpha : (\hat{r},\hat{s}) \subseteq (r,s) \text{ or } (r,s) \subseteq (\hat{r},\hat{s})\}$$

which has the property that $p[T_\alpha^{(r,s)}] = \{y\}$. But by absoluteness of well-founded trees, it must be that there exists $(u,y_0) \in [T_\alpha^{(r,s)}]$ with $(u,y_0) \in L$. But then $y_0 = y \in L$ which is a contradiction. This proves the claim.

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Since $T_{\alpha_0+1} = T_{\alpha_0}$, it follows that for every $(r,s) \in T_{\alpha_0}$ there exist

$$(r_0,s_0),(r_1,s_1) \in T_{\alpha_0}$$

such that $(r_0,s_0),(r_1,s_1)$ extend (r,s) and s_0 and s_1 are incompatible. This allows us to build by induction (working in L):

$$\langle (r_\sigma, s_\sigma) : \sigma \in 2^{<\omega} \rangle$$

with $(r_\sigma,s_\sigma) \in T_{\alpha_0}$ and for each $\sigma \in 2^{<\omega}$ $(r_{\sigma 0}, s_{\sigma 0}),(r_{\sigma 1},s_{\sigma 1})$ extend (r_σ,s_σ) and $s_{\sigma 0}$ and $s_{\sigma 1}$ are incompatible. For any $q \in 2^\omega$ define

$$x_q = \bigcup_{n<\omega} r_{q\restriction n} \text{ and } y_q = \bigcup_{n<\omega} s_{q\restriction n}.$$

Then we have that $(x_q,y_q) \in [T_{\alpha_0}]$ and therefore $P = \{y_q : q \in 2^\omega\}$ is a perfect set such that

$$P \subseteq p[T_{\alpha_0}] \subseteq p[T] = A$$