Part II
Analytic sets

17 Analytic sets

Analytic sets were discovered by Souslin when he encountered a mistake of Lebesgue. Lebesgue had erroneously proved that the Borel sets were closed under projection. I think the mistake he made was to think that the countable intersection commuted with projection. A good reference is the volume devoted to analytic sets edited by Rogers [91]. For the more classical viewpoint of operation-A, see Kuratowski [57]. For the whole area of descriptive set theory and its history, see Moschovakis [87].

Definition. A set $A \subseteq \omega^\omega$ is Σ^1_1 iff there exists a recursive

$$R \subseteq \bigcup_{n\in\omega} (\omega^n \times \omega^n)$$

such that for all $x \in \omega^\omega$

$$x \in A \text{ iff } \exists y \in \omega^\omega \ \forall n \in \omega \ \ R(x \restriction n, y \restriction n).$$

A similar definition applies for $A \subseteq \omega$ and also for $A \subseteq \omega \times \omega^\omega$ and so forth. For example, $A \subseteq \omega$ is Σ^1_1 iff there exists a recursive $R \subseteq \omega \times \omega^{<\omega}$ such that for all $m \in \omega$

$$m \in A \text{ iff } \exists y \in \omega^\omega \ \forall n \in \omega \ \ R(m, y \restriction n).$$

A set $C \subseteq \omega^\omega \times \omega^\omega$ is Π^0_1 iff there exists a recursive predicate

$$R \subseteq \bigcup_{n\in\omega} (\omega^n \times \omega^n)$$

such that

$$C = \{(x, y) : \forall n \ R(x \restriction n, y \restriction n)\}.$$

That means basically that C is a recursive closed set.

The Π classes are the complements of the Σ's and Δ is the class of sets which are both Π and Σ. The relativized classes, e.g. $\Sigma^1_1(x)$ are obtained by allowing R to be recursive in x, i.e., $R \leq_T x$. The boldface classes, e.g., $\utilde{\Sigma}^1_1$, $\utilde{\Pi}^1_1$, are obtained by taking arbitrary R's.

Lemma 17.1 *$A \subseteq \omega^\omega$ is Σ^1_1 iff there exists set $C \subseteq \omega^\omega \times \omega^\omega$ which is Π^0_1 and*

$$A = \{x \in \omega^\omega : \exists y \in \omega^\omega \ \ (x, y) \in C\}.$$

Lemma 17.2 *The following are all true:*

1. *For every $s \in \omega^{<\omega}$ the basic clopen set $[s] = \{x \in \omega^\omega : s \subseteq x\}$ is Σ^1_1,*