15 The random real model

In this section we consider the question of Borel orders in the random real model. We conclude with a few remarks about perfect set forcing.

A set $X \subseteq 2^\omega$ is a *Sierpiński set* iff X is uncountable and for every measure zero set Z we have $X \cap Z$ countable. Note that by Mahlo's Theorem 10.2 we know that under CH Sierpiński sets exists. Also it is easy to see that in the random real model, the set of reals given by the generic filter is a Sierpiński set.

Theorem 15.1 *(Poprougenko [89]) If X is Sierpiński, then* $\text{ord}(X) = 2$.

proof:

For any Borel set $B \subseteq 2^\omega$ there exists an F_σ set with $F \subseteq B$ and $B \setminus F$ measure zero. Since X is Sierpiński $(B \setminus F) \cap X = F_0$ is countable, hence F_σ. So

$$B \cap X = (F \cup F_0) \cap X.$$

■

I had been rather hoping that every uncountable separable metric space in the random real model has order either 2 or ω_1. The following result shows that this is definitely not the case.

Theorem 15.2 *Suppose $X \in V$ is a subspace of 2^ω of order α and G is measure algebra 2^κ-generic over V, i.e. adjoin κ many random reals.*

Then $V[G] \models \alpha \leq \text{ord}(X) \leq \alpha + 1$.

The result will easily follow from the next two lemmas.

Presumably, $\text{ord}(X) = \alpha$ in $V[G]$, but I haven't been able to prove this. Fremlin's proof (Theorem 13.4) having filled up one such missing gap, leaving this gap here restores a certain cosmic balance of ignorance.[5]

Clearly, by the usual ccc arguments, we may assume that $\kappa = \omega$ and G is just a random real. In the following lemmas boolean values $[\![\theta]\!]$ will be computed in the measure algebra $\mathbb{B}$ on 2^ω. Let μ be the usual product measure on 2^ω.

Lemma 15.3 *Suppose ϵ a real, $b \in \mathbb{B}$, and $\overset{\circ}{U}$ the name of a $\mathbf{\Pi}^0_\alpha$ subset of 2^ω in $V[G]$. Then the set*

$$\{x \in 2^\omega : \mu(b \wedge [\![\check{x} \in \overset{\circ}{U}]\!]) \geq \epsilon\}$$

is $\mathbf{\Pi}^0_\alpha$ in V.

proof:

The proof is by induction on α.

Case $\alpha = 1$.

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[5] All things I thought I knew; but now confess, the more I know I know, I know the less.- John Owen (1560-1622)