11 Martin-Solovay Theorem

In this section we the theorem below. The technique of proof will be used in the next section to produce a boolean algebra of order ω_1.

Theorem 11.1 *(Martin-Solovay [72]) The following are equivalent for an infinite cardinal κ:*

1. MA_κ, *i.e., for any poset $\mathbb{P}$ which is ccc and family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G with $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$*

2. *For any ccc σ-ideal I in* $\mathrm{Borel}(2^\omega)$ *and $\mathcal{I} \subset I$ with $|\mathcal{I}| < \kappa$ we have that*
$$2^\omega \setminus \bigcup \mathcal{I} \neq \emptyset.$$

Lemma 11.2 *Let $\mathbb{B} = \mathrm{Borel}(2^\omega)/I$ for some ccc σ-ideal I and let $\mathbb{P} = \mathbb{B} \setminus \{0\}$. The following are equivalent for an infinite cardinal κ:*

1. *for any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ there exists a $\mathbb{P}$-filter G with $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$*

2. *for any family $\mathcal{F} \subseteq \mathbb{B}^\omega$ with $|\mathcal{F}| < \kappa$ there exists an ultrafilter U on $\mathbb{B}$ which is $\mathcal{F}$-complete, i.e., for every $\langle b_n : n \in \omega \rangle \in \mathcal{F}$*
$$\sum_{n \in \omega} b_n \in U \textit{ iff } \exists n\, b_n \in U$$

3. *for any $\mathcal{I} \subset I$ with $|\mathcal{I}| < \kappa$*
$$2^\omega \setminus \bigcup \mathcal{I} \neq \emptyset$$

proof:

To see that (1) implies (2) note that for any $\langle b_n : n \in \omega \rangle \in \mathbb{B}^\omega$ the set
$$D = \{p \in \mathbb{P} : p \leq -\sum_n b_n \text{ or } \exists n\, p \leq b_n\}$$
is dense. Note also that any filter extends to an ultrafilter.

To see that (2) implies (3) do as follows. Let H_γ stand for the family of sets whose transitive closure has cardinality less than the regular cardinal γ, i.e. they are hereditarily of cardinality less than γ. The set H_γ is a natural model of all the axioms of set theory except possibly the power set axiom, see Kunen [54]. Let M be an elementary substructure of H_γ for sufficiently large γ with $|M| < \kappa$, $I \in M, \mathcal{I} \subseteq M$.

Let $\mathcal{F}$ be all the ω-sequences of Borel sets which are in M. Since $|\mathcal{F}| < \kappa$ we know there exists U an $\mathcal{F}$-complete ultrafilter on $\mathbb{B}$. Define $x \in 2^\omega$ by the rule:
$$x(n) = i \text{ iff } [\{y \in 2^\omega : y(n) = i\}] \in U.$$