8 Boolean algebras

In this section we consider the length of Borel hierarchies generated by a subset of a complete boolean algebra. We find that the generators of the complete boolean algebra associated with α-forcing generate it in exactly $\alpha+1$ steps. We start by presenting some background information.

Let $\mathbb{B}$ be a *cBa*, i.e, complete boolean algebra. This means that in addition to being a boolean algebra, infinite sums and products, also exist; i.e., for any $C \subseteq \mathbb{B}$ there exists b (denoted $\sum C$) such that

1. $c \leq b$ for every $c \in C$ and

2. for every $d \in \mathbb{B}$ if $c \leq d$ for every $c \in C$, then $b \leq d$.

Similarly we define $\prod C = -\sum_{c \in C} -c$ where $-c$ denotes the complement of c in $\mathbb{B}$.

A partial order $\mathbb{P}$ is *separative* iff for any $p, q \in \mathbb{P}$ we have

$$p \leq q \text{ iff } \forall r \in \mathbb{P}(r \leq p \text{ implies } q, r \text{ compatible}).$$

Theorem 8.1 *(Scott, Solovay see [43]) A partial order $\mathbb{P}$ is separative iff there exists a cBa $\mathbb{B}$ such that $\mathbb{P} \subseteq \mathbb{B}$ is dense in $\mathbb{B}$, i.e. for every $b \in \mathbb{B}$ if $b > 0$ then there exists $p \in \mathbb{P}$ with $p \leq b$.*

It is easy to check that the α-forcing $\mathbb{P}$ is separative (as long as $\mathcal{B}$ is infinite): If $p \not\leq q$ then either

1. t_p does not extend t_q, so there exists s such that $t_q(s) = B$ and either s not in the domain of t_p or $t_p(s) = C$ where $C \neq B$ and so in either case we can find $r \leq p$ with r, q incompatible, or

2. F_p does not contain F_q, so there exists $(s, x) \in (F_q \setminus F_p)$ and we can either add $(s\hat{\ }n, x)$ for sufficiently large n or add $t_r(s\hat{\ }n) = B$ for some sufficiently large n and some $B \in \mathcal{B}$ with $x \in B$ and get $r \leq p$ which is incompatible with q.

The elegant (but as far as I am concerned mysterious) approach to forcing using complete boolean algebras contains the following facts:

1. for any sentence θ in the forcing language

   $$[\![\, \theta \,]\!] = \sum\{b \in \mathbb{B} : b \Vdash \theta\} = \sum\{p \in \mathbb{P} : p \Vdash \theta\}$$

   where $\mathbb{P}$ is any dense subset of $\mathbb{B}$,

2. $p \Vdash \theta$ iff $p \leq [\![\, \theta \,]\!]$,

3. $[\![\, \neg\theta \,]\!] = -[\![\, \theta \,]\!]$,

4. $[\![\, \theta \wedge \psi \,]\!] = [\![\, \theta \,]\!] \wedge [\![\, \psi \,]\!]$,