6 Generic G_δ

It is natural[4] to ask

"What are the possibly lengths of Borel hierarchies?"

In this section we present a way of forcing a generic G_δ.

Let X be a Hausdorff space with a countable base $\mathcal{B}$. Consider the following forcing notion.

$p \in \mathbb{P}$ iff it is a finite consistent set of sentences of the form:

1. "$B \subseteq \overset{\circ}{U}_n$" where $B \in \mathcal{B}$ and $n \in \omega$, or
2. "$x \notin \overset{\circ}{U}_n$" where $x \in X$ and $n \in \omega$, or
3. "$x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$" where $x \in X$.

Consistency means that we cannot say that both "$B \subseteq \overset{\circ}{U}_n$" and "$x \notin \overset{\circ}{U}_n$" if it happens that $x \in B$ and we cannot say both "$x \notin \overset{\circ}{U}_n$" and "$x \in \bigcap_{n<\omega} \overset{\circ}{U}_n$". The ordering is reverse inclusion. A $\mathbb{P}$ filter G determines a G_δ set U as follows: Let

$$U_n = \bigcup \{B \in \mathcal{B} : \text{“}B \subseteq \overset{\circ}{U}_n\text{”} \in G\}.$$

Let $U = \bigcap_n U_n$. If G is $\mathbb{P}$-generic over V, a density argument shows that for every $x \in X$ we have that

$$x \in U \text{ iff “}x \in \bigcap_{n<\omega} \overset{\circ}{U}_n\text{”} \in G.$$

Note that U is not in V (as long as X is infinite). For suppose $p \in \mathbb{P}$ and $A \subseteq X$ is in V is such that

$$p \Vdash \overset{\circ}{U} = \check{A}.$$

Since X is infinite there exist $x \in X$ which is not mentioned in p. Note that $p_0 = p \cup \{\text{“}x \in \bigcap_{n<\omega} \overset{\circ}{U}_n\text{”}\}$ is consistent and also $p_1 = p \cup \{\text{“}x \notin \overset{\circ}{U}_n\text{”}\}$ is consistent for all sufficiently large n (i.e. certainly for U_n not mentioned in p.) But $p_0 \Vdash x \in \overset{\circ}{U}$ and $p_1 \Vdash x \notin \overset{\circ}{U}$, and since x is either in A or not in A we arrive at a contradiction.

In fact, U is not F_σ in the extension (assuming X is uncountable). To see this we will first need to prove that $\mathbb{P}$ has ccc.

Lemma 6.1 *$\mathbb{P}$ has ccc.*

proof:

Note that p and q are compatible iff $(p \cup q) \in \mathbb{P}$ iff $(p \cup q)$ is a consistent set of sentences. Recall that there are three types of sentences:

[4] 'Gentlemen, the great thing about this, like most of the demonstrations of the higher mathematics, is that it can be of no earthly use to anybody.' -Baron Kelvin