§8. Embeddings of K

In this section we prove some general theorems concerning embeddings of K. We also use these theorems and the ideas behind them to give another proof that if there is a strongly compact cardinal, then there is an inner model with a Woodin cardinal.

In his work on the core model for sequences of measures ([M1] , [M ?]), Mitchell has shown that if there is no inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$, then for any universal weasel M there is an elementary $j : K \to M$; moreover, for any weasel M and elementary $j : K \to M$, j is the iteration map associated to some (linear) iteration of K. Thus the class of embeddings of K is precisely the class of iteration maps, and the class of range models for such embeddings is precisely the class of universal weasels. It follows at once that if there is no inner model satisfying $\exists\kappa(o(\kappa) = \kappa^{++})$, then any $j : K \to K$ is the identity; that is, K is "rigid".

Mitchell's results extend the original Dodd-Jensen theorem ([DJ1]) that if there is no inner model with a measurable cardinal, then whenever $j : K \to M$ is elementary, $M = K$ and $j =$ identity. The authors of [DJKM] strengthen Mitchell's results by weakening their non-large-cardinal hypothesis to "There is no inner model with a strong cardinal". We shall also prove such a strengthening of Mitchell's results, in Theorem 8.13 below.

The situation becomes more complicated once one gets past strong cardinals. We shall see that it is consistent with "There is no inner model having two strong cardinals" that there is a universal weasel which is not an iterate of K, and an elementary $j : K \to M$ which is not an iteration map. Assuming only that there is no inner model with a Woodin cardinal, however, we can still show that K is rigid. Using this fact, we can characterize K as the unique universal weasel which is elementarily embeddable in all universal weasels. We shall also show that if $j : K \to M$, where M is $\Omega+1$ iterable, and $\mu = \text{crit}(j)$, then $P(\mu)^K = P(\mu)^M$. We shall assume throughout this section that K^c satisfies "There are no Woodin cardinals", so that Ω is A_0-thick in K^c and $K = \text{Def}(K^c, A_0)$. Since we need only consider S-thick sets for $S = A_0$, we make the following definition.

Definition 8.1. *We say Γ is thick in W iff Γ is A_0-thick in W. Similarly, W has the hull (resp. definability) property at α iff W has the A_0-hull (resp. definability) property at α. Finally, $Def(W) = Def(W, A_0)$.*

We begin by showing that for any $\alpha < \Omega$, one can generate a witness that $\mathcal{J}_\alpha^K$ is A_0-sound from K by taking ultrapowers by the order zero measures at each measurable cardinal κ of K such that $\alpha < \kappa < \Omega$. The key to this result is the following.

Lemma 8.2. *Let W be an $\Omega+1$ iterable weasel which has the hull property at all $\alpha < \Omega$; then there is an iteration tree $\mathcal{T}$ on K with last model $\mathcal{M}_\theta^\mathcal{T} = W$, and such that*