We shall now say that P is reducible to Q if

$$P(\vec{\alpha},\vec{x}) \leftrightarrow Q(\lambda y G_1(y,\vec{\alpha},\vec{x}),\dots,\lambda y G_m(y,\vec{\alpha},\vec{x}),F_1(\vec{\alpha},\vec{x}),\dots,F_k(\vec{\alpha},\vec{x}))$$

where $G_1,\dots,G_m,F_1,\dots,F_k$ are total and recursive.

19.3. PROPOSITION. If P is Π^1_n and Q is reducible to P, then P is Π^1_n; and similarly with Σ^1_n or Δ^1_n in place of Π^1_n. □

The analogue of the table in §12 is the following table.

P,Q	$\neg P$	$P \vee Q$	$P \,\&\, Q$	$\forall\alpha P$	$\exists\alpha P$	QxP
Π^1_n	Σ^1_n	Π^1_n	Π^1_n	Π^1_n	Σ^1_{n+1}	Π^1_n
Σ^1_n	Π^1_n	Σ^1_n	Σ^1_n	Π^1_{n+1}	Σ^1_n	Σ^1_n
Δ^1_n	Δ^1_n	Δ^1_n	Δ^1_n	Π^1_n	Σ^1_n	Δ^1_n

It is proved and used in the same way as the earlier table.

The classification of analytical relations into the Π^1_n and Σ^1_n relations is called the analytical hierarchy.

19.4. ANALYTICAL ENUMERATION THEOREM. For every n, m, and k, there is a Π^1_n $(m,k+1)$–ary function which enumerates the class of Π^1_n (m,k)–ary relations; and similarly with Σ^1_n for Π^1_n.

Proof. Suppose, for example,we want to enumerate the Π^1_2 $(1,1)$–ary relations. Every such relation R is of the form $\forall\alpha\exists\beta P$ where P is Π^0_1 by the remarks after 19.1. Thus if Q is Π^0_1 and enumerates the Π^0_1 $(3,1)$–ary relations, then $\forall\alpha\exists\beta Q(\alpha,\beta,\gamma,x,e)$ is the desired enumerating function. □

19.5. ANALYTICAL HIERARCHY THEOREM. For each n, there is a Π^1_n set which is not Σ^1_n, hence not Π^1_k or Σ^1_k for any $k < n$. The same holds with Π^1_n and Σ^1_n interchanged.

Proof. As in the arithmetical case. □

20. The Projective Hierarchy

The results of the last section can be relativized to a class Φ of total functions of number variables. A particularly interesting case is that in which Φ